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Mathematics Curriculum: Paving the Road to Student Learning

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RCML History

The Research Council on Mathematics Learning, formerly The Research Council for Diagnostic and Prescriptive Mathematics, grew from a seed planted at a 1974 national conference held at Kent State University. A need for an informational sharing structure in diagnostic, prescriptive, and remedial mathematics was identified by James W. Heddens. A group of invited professional educators convened to explore, discuss, and exchange ideas especially in regard to pupils having difficulty in learning mathematics. It was noted that there was considerable fragmentation and repetition of effort in research on learning deficiencies at all levels of student mathematical development. The discussions centered on how individuals could pool their talents, resources, and research efforts to help develop a body of knowledge. The intent was for teams of researchers to work together in collaborative research focused on solving student difficulties encountered in learning mathematics.

Specific areas identified were:

1. Synthesize innovative approaches.
2. Create insightful diagnostic instruments.
3. Create diagnostic techniques.
4. Develop new and interesting materials.
5. Examine research reporting strategies.

As a professional organization, the **Research Council on Mathematics Learning (RCML)** may be thought of as a vehicle to be used by its membership to accomplish specific goals. There is opportunity for everyone to actively participate in **RCML**. Indeed, such participation is mandatory if **RCML** is to continue to provide a forum for exploration, examination, and professional growth for mathematics educators at all levels.

The Founding Members of the Council are those individuals that presented papers at one of the first three National Remedial Mathematics Conferences held at Kent State University in 1974, 1975, and 1976.

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K-16 Students' Construction of Mathematics

Developing Capacity and Agency in Mathematics Teaching and Learning

CONCEPTUAL VERSUS PROCEDURAL APPROACHES TO ORDERING FRACTIONS

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This paper reports the performance of 30 rising seventh-grade girls on a task that involved ordering four fractions from least to greatest. Less than three-fifths attained correct answers. The performance gap was widest between students who attended Title I schools and those who did not, the latter being much more likely to attain correct answers. Participants tended to use procedural and conceptual approaches equally, but conceptual approaches were more successful.

Knowledge of fractions is foundational to many areas of mathematics learning, including algebra, proportional reasoning, and probability (Clarke & Roche, 2009; Fennell, 2007; Siegler & Pyke, 2013). One important skill is an ability to determine the relative size of fractions. This involves using an understanding of fraction magnitude to compare two fractions to decide whether they are equivalent or which is greater or less, or to order three or more fractions from smallest to largest or vice versa. Comparing fraction size appears in the Common Core State Standards for grades three and four (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010), but it continues to be an important mathematics skill thereafter. In this paper, we report the performance of 30 girls in the summer before their seventh-grade year ("rising seventh graders") on a fraction-ordering task.

Related Literature

Student Challenges in Learning Fractions

Despite the importance of fraction knowledge, students struggle to learn fraction concepts, which includes fraction comparison (Clarke & Roche, 2009; Pantziara & Philippou, 2012; Siegler & Pyke, 2013; Sprute & Temple, 2011). For example, on the 2007 National Assessment of Educational Progress, only half of U.S. eighth-grade students correctly chose the ordered set 2/7, 1/2, and 5/9 as that which appeared from least to greatest from among five multiple choice options of three fractions each (Institute of Education Sciences, 2007). Further, a calculator was available for this item.

Some factors that contribute to students' weak performance with fractions include inappropriate transfer of whole-number ideas to fractions (e.g., larger numbers mean greater magnitude) and a focus on individual fraction components (numerator, denominator) rather than a fraction as a single entity (Clarke & Roche, 2009; Lamon, 2012; Siegler & Pyke, 2013). Pantziara and Philippou (2012) note that students perform most poorly on the fraction subconstruct of "measure," in which students identify a fraction as a point on a number line. This ability is important to understanding the relative size of fractions, which in turn helps students view a fraction as a single number, thus suggesting the need for greater use of the number line for fraction investigations (Fennell, 2007; Pantziara & Philippou, 2012; Sprute & Temple, 2011). Pantziara & Philippou (2012) note that the measure subconstruct appears in mathematics textbooks and instruction less frequently than the dominant part-whole subconstruct.

One key source of struggle with fraction magnitude is weak knowledge of the role of the numerator and denominator. Students might gauge fraction magnitude by the size of either the numerator or the denominator rather than integrating the two meaningfully (Meert, Grégoire, & Noël, 2010; Siegler & Pyke, 2013; Sprute & Temple, 2011), and whole-number interference can lead students to think larger denominators mean larger fractions (Pantziara & Philippou, 2012).

Role of Conceptual and Procedural Knowledge

Conceptual knowledge involves understanding information in a meaningful, relational way, such as being able to explain why or how something works or making sense of it in a real-world context. Whereas procedural knowledge is an ability to execute standard steps to solve a task; as in implementing an algorithm (e.g., Hallett et al., 2010; Tularam & Hulsman, 2013). Many educators and researchers note that both are important to successful mathematics learning (Gabriel, Coché, Szucs, Carette, Rey, & Content, 2012; Hallett, Nunes, & Bryant, 2010; Pantziara & Philippou, 2012). The distinction between using conceptual and procedural approaches seems to be particularly apparent in relation to fractions (Hallett et al., 2010).

Suggested Instructional Approaches for Improving Fraction Understanding

It is imperative that students learn the meaning of the fraction numerator and denominator and their relationship to each other in a holistic or unified manner (Clarke & Roche, 2009; Siegler & Pyke, 2013). Some instructional strategies recommended for improving students' understanding of fractions are those that allow students to use and make visual representations of fractions, partition objects and drawings into equal-sized pieces, visualize fraction concepts (e.g., physical and pictorial representations), learn to see fractions as a single entity and locate them on a number line that includes whole numbers, and develop more than one strategy for solving fraction tasks, including number-sense approaches such as use of benchmarks, and use these approaches in active, engaging, collaborative endeavors (Bray & Abreu-Sanchez, 2010; Gabriel et al., 2012; Lamon, 2012; Pantziara & Philippou, 2012; Sprute & Temple, 2011).

Research Purpose and Method

The purpose of this investigation was to examine rising seventh-grade girls' performance on ordering four fractions from least to greatest and the broad methods (conceptual or procedural) used to do so. For exploratory purposes, we further examined the girls' performance by the demographic variables of race/ethnicity, community type (urban/suburban versus rural), and socioeconomic status (SES) measured at both the family and school levels.

Participants

Of the 30 rising seventh-grade girls from across Northern Nevada who participated in this study, 22 (73.3%) were from urban/suburban communities and 8 (26.7%) were from rural communities, and 21 (70.0%) were White, 3 (10.0%) were Latina, 2 (6.7%) were Black/African American, 2 (6.7%) were American Indian/Alaska Native, and 2 (6.7%) were of mixed race/ethnicity. In terms of family SES, 17 girls (56.7%) were classified as medium/high and 13 (43.3 %) as low. Regarding school SES: 19 (63.3%) girls had not attended a Title I school the previous year, whereas 11 (36.7%) had attended a Title I school. At the time the data were collected, all girls were attending a one-week, summer, residential math and technology camp.

Assessment Item

The assessment item participants solved individually on paper without a calculator was: Order the fractions from least to greatest: 1/1, 1/3, 5/8, 1/12.

Data Analysis

In addition to scoring each problem as correct or incorrect, the general approach used to answer the problem was classified as conceptual, procedural, or indeterminate. Due to the low number of participants in this study, no statistical analyses were conducted. Therefore, we report the results as numbers and percents to be examined for exploratory purposes only.

Results

Conceptual Versus Procedural Approaches

Of the 30 participants who completed this problem, 13 (43.3%) used procedural approaches, 12 (40.0%) used conceptual approaches, and 5 (16.7%) used mental methods that could not be determined. More than half of all students attained a correct answer for the problem, but the proportion was highest for those who used conceptual methods. (See Table 1.)

Table 1. *Problem Accuracy by Major Solution Strategy*

Solution Strategy	Accuracy	
	Correct	Incorrect
Procedural	7 (54.0%)	6 (46.0%)
Conceptual	8 (66.7%)	4 (33.3%)
Indeterminate (Mental)	2 (40.0%)	3 (60.0%)
Totals	17 (56.7%)	13 (43.3%)

Correct problem solutions ranged among demographic groups from 73.7% for students who did not attend Title I schools to 27.3% for students who attended Title I schools. The difference between these two school SES categories was the widest among demographic groups assessed, with groups formed by race/ethnicity, family SES, and community type being separated by narrower margins. Among the three demographic groups with smaller performance gaps, racial minorities outperformed White students, and students from low family SES backgrounds were somewhat more successful than students of middle/high family SES, as were urban/suburban students compared with rural students. Students more likely to use conceptual solution strategies were White (compared with racial minorities), middle/high (versus low) family SES, from a non-Title I (versus Title I) school, and urban/suburban (compared with rural).

Specific Solution Strategies

The two most common correct approaches were converting the fractions to equivalent fractions (a procedural approach) and making a drawing (a conceptual approach). Specifically, 6 involved converting the fractions to equivalent fractions with a common denominator (3 of these did so only for two of the four fractions, 1/3 and 5/8), and 5 involved making a drawing of rectangles and/or circles partitioned into equal parts with some shaded. Two other correct methods involved use of benchmarks and converting the fractions to decimals. The most common incorrect approach was to reverse the order of the middle two fractions, and the next most common was to reverse the order of all four fractions (presented greatest to least).

Of the 6 of 25 identifiable strategies that involved use of a drawing, 5 resulted in a correct answer and 1 an incorrect answer. These drawings were exclusively area and length models. No student used a number line, which is another type of length model, or a set model. Of the three students who used written explanation to address the size of the fractional parts, none gave evidence of blending the numerator and denominator into a meaningful single numeric value.

Discussion

Overall, student performance on this fraction task tends to mirror student performance in other research studies (e.g., Institute of Education Sciences, 2007) in that students do not show strong performance comparing and ordering fractions. Students used conceptual and procedural approaches equally, but conceptual approaches were somewhat more successful. This aligns with perspectives in the field that consider conceptual approaches particularly important, while acknowledging the key role of procedural knowledge (e.g., Hallett et al., 2010; Siegler & Pyke, 2013). One concern with procedural approaches is whether students also try to make conceptual sense of their work. For example, one participant converted all four fractions in this task to decimals. However, she made an error on one, attaining 0.83 instead of 0.083 for 1/12. The fact that she did not notice this major mistake is a lingering potential issue with procedural strategies.

Similar to previous research, students tended to use a limited number of success-oriented approaches, in particular, use of equivalent fractions and drawings. In 6 of the 25 identifiable strategies, participants used efficient methods involving drawings or conversions to decimals or equivalent fractions only for those fractions that seemed to challenge them the most (here, 1/3 and 5/8). Drawings used as aids to solve the problem exclusively employed part-whole models, which have been found to dominate mathematics textbooks (Pantziara & Philippou, 2012). Partitioning visual (e.g., area) models is considered an important aid to comparing fractions, although student difficulties with doing so have been noted (Lamon, 2012). No student attempted to locate the fractions on a number line or to use a set model, and only one student used benchmarks by comparing the fractions to one-half or one.

Three students appeared to know that denominator size relates to fraction magnitude, but they tended not to integrate that information with numerator size to form a single value, reflecting a problematic focus on individual fraction components (e.g., Lamon, 2012; Siegler & Pyke, 2013). In this same vein, two students used the inaccurate and unsuccessful approach termed “gap thinking” that involves comparing the difference between the numerator and denominator of individual fractions (Clarke & Roche, 2009).

The most common error in answers, regardless of specific strategy used, was to reverse the order of the middle two fractions but correctly place the first and last. This fits Sprute and Temple's (2011) finding of the distance effect in which problem solvers tend to be able to order fractions that are farther apart more successfully than those that are closer together in magnitude. However, the difference between the middle two fractions was not greatly different in size than that of other adjacent pairs, so it may be that participants found it easier to place the most extreme fractions, the two closest to 0 and 1, than the two in the middle. It should also be noted that two denominators and two numerators that differ and are not such that one is a multiple of the other are among pairs that are more difficult to compare in size (Clarke & Roche, 2009). Similarly, no students used the effective strategy known as residual thinking, whereby students determine the amount needed to build up to a whole (cf. Clarke & Roche, 2009). Perhaps this is because this strategy has not been fostered in classroom engagement with fractions or because no fractions in this task evoked this method, as in two fractions that are each one part away from a whole (e.g., 3/4 and 4/5). Thus, the specific fractions used in a task can influence student performance. The less frequent error in which three students presented the answer in reverse order could be due to students misreading or not attending to the problem directions.

The overall performance on this task seems to indicate that students are more likely to achieve correct results when using conceptual approaches or when applying procedural methods that they understand or are well rehearsed. It also appears that students could benefit by developing a greater range of strategies for comparing fraction size. In particular, use of number lines, benchmarks, and set models appear to be used infrequently in textbooks and classroom instruction but would benefit students by expanding their repertoires (Bray & Abreu-Sanchez, 2010; Clarke & Roche, 2009; Lamon, 2012; Pantziara & Philippou, 2012; Sprute & Temple, 2011). All of these methods can promote conceptual thinking, including fraction sense, which is highly important to student success in working with fractions (Lamon, 2012). We thus contend that while student difficulties with fractions might also be conceptual, instructional approaches used for learning them are especially important to student understanding and performance.

Comparisons of performance by race/ethnicity, family SES, school SES, and community type showed the widest discrepancy between participants who did and did not attend a Title I school, potentially indicating that school SES is more influential in a student's education than family SES. Although it is not surprising that urban/suburban students were somewhat more successful than rural students, it is atypical that racial minorities outperformed White students and students from low family SES backgrounds were somewhat more successful than students with higher SES backgrounds (cf. Brown-Jeffy, 2009 and Chiu, 2010). It is uncertain whether this is because the students who solved this problem were attending a mathematics camp and thus already had some degree of interest and/or success in mathematics, making them less representative of the population at large, or whether these individuals who tend to exhibit lower performance in mathematics did not presume successful performance and thus took the task more seriously and exerted more effort and mindfulness. Finally, for all four demographic-group pairs examined, the group that typically shows greater mathematics achievement (White, middle/high family SES, non-Title I school, urban/suburban community residence) was more likely to use conceptual solution strategies. This is worth pursuing further to determine whether this finding holds across larger research samples. If so, it would highlight the importance of expecting all students, including underperformers, to develop and use conceptual approaches to mathematics.

This research supports a number of general findings reported in the literature (e.g., the importance of an emphasis on conceptual understanding) and fraction-specific knowledge (e.g., students' erroneous use of "gap thinking"), but it is particularly useful in highlighting the importance of using a greater variety of instructional approaches in teaching strategies and curricular materials. Further, this study suggests that we look more carefully at the types of approaches different student demographic groups use to solve fraction problems.

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THE GROWTH OF ADDITIVE REASONING WITH SECOND DIFFERENCES

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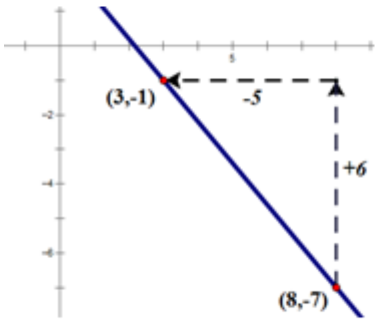
Reasoning about differences between two quantities is difficult and becomes increasingly important in middle grades students' work with integers and algebra. Utilizing a constructivist teaching experiment methodology, we worked with two sixth-grade students over the course of eight teaching sessions on complex additive situations in which students operated on differences of pairs of values to construct a second difference. We describe important changes in the students' ability to construct and reflect on the quantities involved in these situations. We hypothesize that purposeful selection of the context and variation of the number and type of missing quantities promoted student learning.

This study looks at data collected from a larger teaching experiment. In the larger study, we utilized a constructivist teaching experiment methodology (Steffe & Ulrich, 2014) in order to investigate how students' additive reasoning with both signed and unsigned quantities is related their multiplicative reasoning. For the present study, we focused on a sub-question limited to student development of additive reasoning: What behaviors characterize the students' progress towards an assimilatory quantitative structure for second difference problems? Our analysis focuses on our work with two students over the course of eight teaching sessions in developing complex additive situations where the student must work with second-order differences (the difference between the differences between two pairs of numbers). We found three main behavioral changes that characterized the students' progress in developing an assimilatory quantitative structure for these problems: (1) increasing differentiation of language, (2) decreasing conflations of quantities, and (3) developing reversibility of their solution methods.

Rationale

Reasoning about differences between quantities is difficult for students (e.g., Thompson, 1993) and becomes increasingly important in middle grades students' work with algebra. For example, in order to have a conceptual understanding of the slope formula, students must, first, be able to interpret the differences in the left-hand expression of Figure 1 as representing actual quantities—the vertical and horizontal distances between two points on the line. Second, the students must operate on these differences to conceptualize slope as the ratio of these differences. As Piaget (1977/2001) and numerous others,(e.g., Dubinsky, 1991; Sfard, 1991) have pointed out, the ability to conceptualize the result of an operation as a quantity and take it as input for further operation is a complicated process that requires an increasing awareness of one's mental actions.



$$\frac{1-{}^{-}7}{3-8}=\frac{{}^{+}6}{{}^{-}5}=-\frac{6}{5}$$

Figure 1. Calculating slope using differences

We investigate two cases of this process. We borrowed several of our problems from Thompson's (1993) previous work on complex additive situations in which students operate on differences additively. This should be easier for students than the kind of multiplicative comparisons used in the slope calculation (Lamon, 2007). We were also influenced by Cifarelli and Sevim's (2014) use of variations in the amount of information given in a problem in order to engender reflection on quantitative structure. Our work differentiates itself by offering a finer-grained analysis than Thompson's in a different context than Cifarelli and Sevim's work.

Second Difference Problems

For the ease of the reader we provide an example of the type of task that we used to motivate students to operate on first difference quantities. We also provide the terminology and quantitative structure used in our discussion.

> *Two basketball games were played last night. Team A and Team C both won their games. Team C claimed that they had a better game because they beat Team D by 16 more points than Team A beat Team B by. If Team A scored 59 points, Team B scored 44 points and Team D scored 9 points, how many points did Team C score?*

In the task above, the student is provided with four quantities (the difference between win margins along with the scores for Teams A, B and D) that are related by a seven-quantity structure. Team A's win margin (FD1: the available first difference) is the difference of Team A and Team B's scores. Team C's win margin (FD2: the derived first difference) is derived from the sum of Team A's win margin and the difference between win margins (SD: the second difference). Finally, the score of Team C can be determined from the sum of Team C's win margin and Team D's score. Figure 2 shows the quantitative structure that underlies this task.

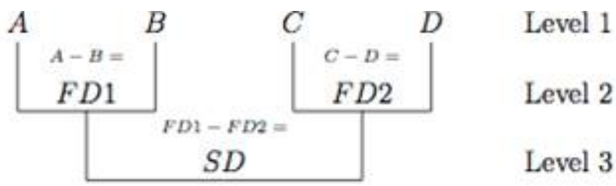


Figure 2. Quantitative Structure of Second Difference Problems

Quantities in the third level can be considered comparisons of two quantities in the second level. Similarly, quantities in the second level compare two of the quantities from the first level. The first level quantities are primary measures in the task. Though the underlying quantitative structure of the tasks does not change, variations included providing students with values for differing quantities (an extraneous or unrelated quantity, the four primary quantities, the three primary quantities, the second difference) and changing the problem setting of the primary quantity (distance run, height, altitude, problems solved and pages read).

Methods

We utilized a constructivist teaching experiment methodology (Steffe & Ulrich, 2014) for the larger study. Because we were investigating the interplay between the construction of additive and multiplicative reasoning at the middle grades, we wanted participants with a variety of reasoning levels that are commonly found at the middle grades. We therefore used initial clinical interviews to identify two sixth-grade students at each of three, hierarchical stages of whole number development as described by Steffe and colleagues (e.g., 2010): maximum construction of a tacitly nested number sequence (TNS), explicitly nested number sequence (ENS), or generalized nested number sequence (GNS). We then began teaching sessions with each pair of students, averaging 16 sessions with each pair over the course of five months. We videotaped each session.

Teaching experiments are characterized by a series of semi-structured clinical interviews with an emphasis not only on determining students' understanding of a problem situation, but also on engendering further understanding of the problem situation in the students. The set of base tasks for each teaching session were developed collaboratively by the researchers to test the boundaries of the students' zones of potential construction (ZPC) and proximal development (Norton & D'Ambrosio, 2008) in specific types of fractional or whole number situations. If the pair either found the mathematics of a situation unproblematic or unbearably hard, the situation would be abandoned or adjusted to a more suitable mathematical complexity both during and between sessions. Therefore, although we engaged all three pairs in complex additive situations, the required complex additive reasoning seemed to be outside of the ZPC of TNS students, and the GNS students transitioned quickly from attempting to make sense of each situation to reassimilating the situation utilizing reified differences. Hence, neither the TNS or GNS pairs yielded data useful for analyzing the constructive mechanisms that lead from making sense of complex additive situations to the ability to assimilate such a situation in terms of reified differences. The student data analyzed for this paper are from Matt and Mary, who were at the ENS stage.

We conducted eight teaching sessions focusing on the second difference problems with Matt and Mary. Initially, during our sessions on second differences, we had hypothesized that the students would be able to assimilate with the quantitative structure of second difference problems after repeated exposure. However, the students made little progress in the first four sessions of exposure to second difference tasks. Therefore, in the sessions that followed we engaged the students in tasks involving variations on the available quantities and experientially tangible first differences.

Recall that our research question for the present study was, "What behaviors characterize the students' progress towards an assimilatory quantitative structure for second difference problems?" In order to answer that question, we transcribed interactions during each of the complex additive tasks. While rewatching the video and looking at student work, one researcher summarized student responses through response time and possible conflations. The other researcher summarized the student reasoning she hypothesized would result in the witnessed behavior. During meetings, the researchers compared their interpretations of different student responses and the important distinctions that each had noted in their analysis. Both researchers then went back and refined their analyses based on the meetings. Additionally, the researchers made hypotheses about each student's trajectory in taking the first differences as quantities that can be operated and reflected on and the second difference as a quantity in its own right. For example, we hypothesized at one point that Mary had to construct the derived first difference before she could reason or talk about it until the very last session. We then went back and checked our hypotheses against the evidence in the videos, transcripts and student work. This resulted in increased precision of coding as we began to differentiate between different types of conflations students' made or differences in the way in which they referred to first and second differences.

Results

We determined through our analysis that there are shifts in the students' use of language, the students' conflation of quantities, and the students' ability to reason reversibly about the additive relationships. Additionally, we link these shifts to an increasing ability to make sense of, remain aware of, reflect on, and assimilate with the additive relationships underlying the second difference problems. Though analysis was performed on both students, due to limitations in space, we will focus on Matt to illustrate our findings.

In the first three teaching sessions, the majority of the students' difficulties centered on their attempts to understand the quantities at play in the context and what information the problem statement was giving about the relationships between these quantities. Much of the discussion centered on the meaning of specific words and phrases and whether similar phrases referred to the same or different types of quantities. For example, in discussing Task 1, the students could at different times interpret the claim "Team A won by 15 more points" as referring to a primary quantity (Team A scored 15 points, which is more points than Team B), first difference (Team A scored 15 more points than Team B), and second difference (Team A won their game by 15 more points than Team C won their game by). Similar subtleties arise when differentiating between heights and elevations, distances run and distances between, and other distinctions that showed up in various contexts. The most pervasive evidence of this issue was that students would often interpret the second difference given in a problem as a difference between primary quantities A and C (see Figure 2). This continued into the second half of the teaching experiment, but occurred less often and students were able to self-correct.

Even after the students appropriately interpreted the quantities in a problem situation, they still lacked the ability to coordinate all the quantities in the quantitative structure, as indicated by conflations students made. When conflating quantities, a student contradictorily interprets a single value as representing two distinct quantities within a solution method. For example, during the third session, Matt conflates a second difference with a first difference: "I added 14 [second difference] and 87 [Sam's sister's height]. Which gave me 101 and I subtracted 193 [Joseph's height] from 101 and I got 92 or something like that, but that's not 14cm smaller than 99." Here Matt interprets the second difference as a first difference between the heights of the two sisters, as indicated by his adding it to Sam's sister's height. Then Matt rejects his own solution because the first differences derived from his solution are not 14cm apart, implying an appropriate interpretation of the 14 as a second difference. The frequency of conflations decreased markedly after the fourth session for Matt and the sixth session for Mary, indicating the increased permanence of the quantitative structure. Recall that the fifth session is when we began utilizing experientially tangible first differences, i.e., a tree's height versus a difference in heights as the first difference. This is also when we started varying the given information in second difference situations so that students were sometimes given all of the primary quantities or were given only the second difference.

As the students were able to retain the quantitative relationships for a given problem, they were increasingly able to reflect on the quantitative structure. For both Matt and Mary, this was apparent in their shift from building up the quantitative relationships each time they did a problem to their eventual ability to reason reversibly. In the first three sessions, the students were often flummoxed by how to figure out the answer to a second difference problem. At these times, the teacher would encourage them to guess a value for the fourth primary quantity and then build up the relationships by calculating first differences and the resulting second difference. During the fourth session, as Matt was using this kind of guess-and-check strategy , he started to make his third relatively arbitrary guess, paused, and said, "the bottom thing [quantity D in Figure 2] has to be 16, no… 11! it has to be 11," which is the correct value for one of the first differences. He was then able to reason his way to the correct answer without making any more guesses. In this case, Matt reflected on the quantitative relationships he built up in order to determine how to reason reversibly with the second difference to determine the missing first difference. Although he had to go through this process in some form in the first six sessions, he no longer relied on multiple guesses in the last two sessions. The fact that he was able to reason reversibly in the last two sessions indicates that he no longer had to build up the quantitative structure in each problem. Mary had a similar transformation, although she was making numerous guesses up until the fifth session. Again, recall that we began using experientially tangible first differences and variations in given information during this fifth session. Mary then had only one instance of guessing in each of the sixth and seventh sessions and reasons reversibly in the last session.

As the students were able to increasingly reflect on the quantitative structure, their ability to communicate about the different quantities improved dramatically. For example, during the eighth session, Matt clearly differentiates between a height ("how tall"), a difference in heights ("11 inches taller"), and a difference of height differences ("9 *more* inches taller"):

> Tommy is *11 inches taller* than his sister and then Arthur is *9 more inches taller* than Tommy is than his sister, so I added 9….to get 20. I subtracted 20 from 68 and got *48 inches is how tall his sister is*. (emphasis added)

In addition to a change in language use, by the eighth session the students were no longer conflating quantities and were able to reason reversibly without first building up the quantitative structure through guess-and-check.

Findings and Implications

We found that when students were used to a specific kind of quantitative relationship, such as a second difference, they were no longer bothered by the inherently subtle and ambiguous language in the various contexts. Therefore as opposed to focusing on precise mathematical language, we recommend providing students with the chance to build up quantitative relationships. This helped these students reflect on the quantitative relationships they were building and communicate about them more effectively. In particular, guess-and-check and variations on which quantities were given were helpful.

We found conflations that were distinct from issues of communication in that they seemed to have a basis in the student's current inability to deal with the level of mathematical complexity in the problem situation. If, as in the case of Matt in the fourth session, the student is able to reason reversibly, indicating that the student can construct the necessary quantitative relationships and reason about them, but loses track of the quantities due to the complexity, we found that varying the tasks by giving more or less information than is needed to determine a single answer helped perturb the students into reflecting more deeply on the quantitative relationships in the situations. If the student is not reasoning reversibly about the situation yet, this may imply that the student is having trouble building up the quantitative relationships in the first place. In this case, as with Mary in the fifth session, problems where the student can build up the situation or is given freedom in choosing initial values may be appropriate.

Here students were additively comparing differences. However, when dealing with rate of change in, we would like to students to be able to think about multiplicatively comparing differences. Research on whether and how experience working with differences as quantities in second difference situations contributes to students' understanding of rate of change and integer operations could be fruitful.

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STRENGTHENING PROSPECTIVE ELEMENTARY TEACHERS' UNDERSTANDING OF FACTORS

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Research on pre-service elementary school teachers' (PSTs') understanding of the multiplicative structure of number shows that PSTs struggle to use prime factorization to identify a number's factors. This study investigates the benefits of a sequence of three instructional tasks aimed at strengthening PSTs' understanding of factor by exploring the relation between a number's prime factorization and its factors. Analysis of written pre- and post-assessments of 69 pre- and in-service elementary and special education teachers shows that the use of these tasks strengthened PSTs' abilities to use prime factorization to identify factors and non-factors of both prime and composite numbers.

Consider the following question: Given the number $N = 2^3 \cdot 3^2 \cdot 5$, which of the following numbers $\{3,\ 7,\ 15,\ 22\}$ are factors of *N*? There are multiple routes to a solution. One approach is to compute the whole number value *N* and use trial division to determine if any of the factor candidates divide *N*. Assuming no computation errors are made, one will discover that 3 and 15 are factors of *N* and 7 and 22 are not factors of *N*. In addition to being both valid and straightforward, the approach is the most commonly learned method taught in schools.

A second approach is available that is more efficient, less prone to computational error, and richer in connections to underlying divisibility concepts. The approach relies on an important result in number theory – the Fundamental Theorem of Arithmetic (FTA). The theorem states that each natural number greater than 1 has a unique prime factorization (ignoring order). That is, no two natural numbers greater than 1 share the same prime factorization. Though seemingly benign, the notion of *uniqueness* is foundational to the field of number theory and provides the basis to the solution method. In this less frequently taught method, one examines the prime factorization of *N* to determine if it includes the prime factorization of 3, 7, 15, or 22. If it does include the prime factorization of any of these numbers, then that number is a factor of *N*. Otherwise, the given number is not a factor of *N*. Consider 3 in our example; since $N = 2^3 \cdot 3^2 \cdot 5 = (3) \cdot (2^3 \cdot 3 \cdot 5)$, *N* can be written as a multiple of 3 and so 3 is a factor of *N*. Similarly, 15 is a factor of *N* because the prime factorization of 15 is $15 = 3 \cdot 5$ and *N* can be expressed as $N = 2^3 \cdot 3^2 \cdot 5 = (3 \cdot 5) \cdot (2^3 \cdot 3)$, a multiple of 15. This approach can also determine non-factors. Since the prime factorization of *N* does not include the prime number 7 and no other prime factorization of *N* exists, 7 cannot be a factor of *N*. Likewise, since the prime factorization of 22 is $22 = 2 \cdot 11$, and since *N*'s prime factorization does not include *both* 2 and 11, we cannot write *N* as a multiple of 22. As a result, 22 is not a factor of *N*.

This introduction is meant to reveal the subtle complexities associated with the use of unique prime factorization guaranteed by the FTA to the analysis of factors. Using prime factorization requires knowledge of prime and composite numbers, the associative and commutative properties of multiplication, and a full understanding of the implications of the uniqueness feature of the FTA. In light of this fact, it should come as no surprise that many pre-service elementary school teachers (PSTs) struggle to make use of prime factorization and uniqueness to identify factors (Zazkis & Gadowsky, 2001). This documented difficulty points to PSTs' limited understandings of the multiplicative structure of the natural numbers. This study represents an attempt to remediate this area of concern by examining the benefits of a set of instructional tasks aimed at strengthening understanding of factors and prime factorization.

The Literature

Prior research has shown that pre-service elementary teachers (PSTs) struggle to understand divisibility concepts. The large majority of research in this area has been conducted by Rina Zazkis and her colleagues. In 1996, they conducted clinical interviews with 21 PSTs and found that 15 of them exhibited limited and procedural understandings of divisibility (Zazkis & Campbell, 1996a). When asked to find the factors of a number expressed in prime factored form, participants typically expressed the need to compute the whole number value of the number and perform long division. Zazkis (1998) corroborated this finding, showing that PSTs relied on empirical verification (e.g., long division or application of divisibility rules) and exhibited little ability to use prime factorization as a tool for reasoning about factors. Zazkis and Gadowsky (2001) framed this finding as PSTs' failure to make use of the *transparent* features of prime factorization. In other words, PSTs struggled to take advantage of the affordances that prime factored representations provide in terms of making certain numerical properties easily identifiable (e.g., the prime factored representation of $N = 2^3 \cdot 3^2 \cdot 5$ makes it transparent that 5 is a factor of *N*). Additional research by Brown et al. (2002) investigated how PSTs' divisibility schemas emerge and found that success using action-oriented strategies (e.g., trial division) may impede their use of prime factorization in completing divisibility tasks. They called for pedagogical interventions that emphasize flexible reasoning with numbers written in prime factored form.

Some studies have characterized the extent of PSTs' knowledge of number theory topics. Zazkis (2005) found that PSTs use negative descriptions to define prime numbers (e.g., "prime numbers 'cannot be divided', 'cannot be factored' or 'wouldn't have/are not having any other factor'" (p. 208)), which may be an obstacle to achieving a robust conceptual understanding of prime number. Other studies have identified PSTs' misconceptions about factors and prime numbers, such as the notion that bigger numbers have more factors or that prime numbers are small (Zazkis & Campbell, 1996b; Zazkis & Gadowsky, 2001). Researchers have also noted that PSTs tend to have an easier time identifying factors than non-factors, and are better able to recognize prime factors than composite factors (Zazkis & Campbell, 1996a, 1996b). Zazkis and Campbell (1996b) noted that PSTs' difficulty with identifying non-factors may be due to a lack of appreciation for the uniqueness feature of the FTA: "Whereas the *existence* of prime decomposition may be taken for granted, the *uniqueness* of prime decomposition appears to be counterintuitive and often a possibility of different prime decompositions is assumed" (p. 217).

Only a handful of studies have examined the efficacy of interventions aimed at improving PSTs' knowledge of divisibility and prime factorization. Feldman (2012) implemented a set of paper-and-pencil number theory tasks focused on the use of prime factorization with 59 pre-service elementary teachers. He found that their ability to identify factors and non-factors, as well as solve greatest common factor (GCF) and least common multiple (LCM) problems improved significantly. Also, Sinclair et al. (2004) and Liljedahl et al. (2006) used a computer applet that provided an interactive array where students could explore factors and multiples using a visual representational model in a laboratory setting. The researchers found that the combined effects of *visualization* and *experimentation* led to a more robust understanding of the multiplicative structure of the natural numbers, primes, composites, evens and odds.

Methodology

A classroom intervention was conducted at two large universities in the United States. Participants (n = 69) were undergraduate and graduate students enrolled in mathematics content courses for both pre-service and in-service elementary and special education teachers. Participants were asked to complete a pre-test before the start of the intervention and a post-test approximately two weeks after the conclusion of the intervention. The intervention, which lasted approximately three weeks, consisted of three in-class lessons and two out-of-class homework assignments. The goal of the intervention was for both instructors (the two authors) to facilitate participants' construction of their own understanding of prime factorization and divisibility concepts. In this role, each instructor observed participants' work in small groups of size three or four and only interrupted to ask guiding and probing questions. During whole-class discussions, each instructor encouraged participants to explain and justify their own mathematical thinking and rarely provided solutions themselves.

In Lesson 1, participants use factor trees and various factorizations of the same number to make sense of the Fundamental Theorem of Arithmetic and to recognize that although every counting number (greater than 1) has a unique *prime factorization* it may have several different *factorizations*. In Homework 1, participants fill a 10-by-10 array labeled from 1-100 with the prime factorization of each counting number using any method they choose. Participants then identify patterns in their array. In Lesson 2, participants use their arrays from Homework 1 to find factors of specific numbers in both whole number and prime-factored form (see **Figure 1** below). They discuss their conjectures for how a number's factors are related to its prime factorization. In Lesson 3, participants solve problems in order to develop a general rule for finding the number of factors using the number's prime factorization (i.e., $p_1^{n_1} \cdot p_2^{n_2} \cdot p_3^{n_3} \cdot \ldots \cdot p_m^{n_m}$ has $(n_1+1)(n_2+1)(n_3+1)\ldots(n_m+1)$ factors). Homework 2 is the final assignment of the intervention and serves to reinforce the discoveries made in Lessons 1-3.

Each participant completed written pre- and post-tests prior to and following the intervention. Three identical question types were used, but numbers were changed for pre- and post-tests in problems 1 and 2. Table 1 lists each test question. Across all questions, students were asked to show their work or provide their reasoning. Test scoring was conducted using a researcher-developed rubric. Inter-rater reliability analysis of the rubric was conducted. Both authors independently scored the same set of data (21.7% of the data, or 15 of 69 pre- and post-tests). Analysis of the results of the independent scoring revealed 82.5% agreement. Discrepancies in scoring were resolved via discussion and rubric clarification until 100% agreement was achieved. Once reliability had been established the remaining data were divided equally between the two authors and scored separately.

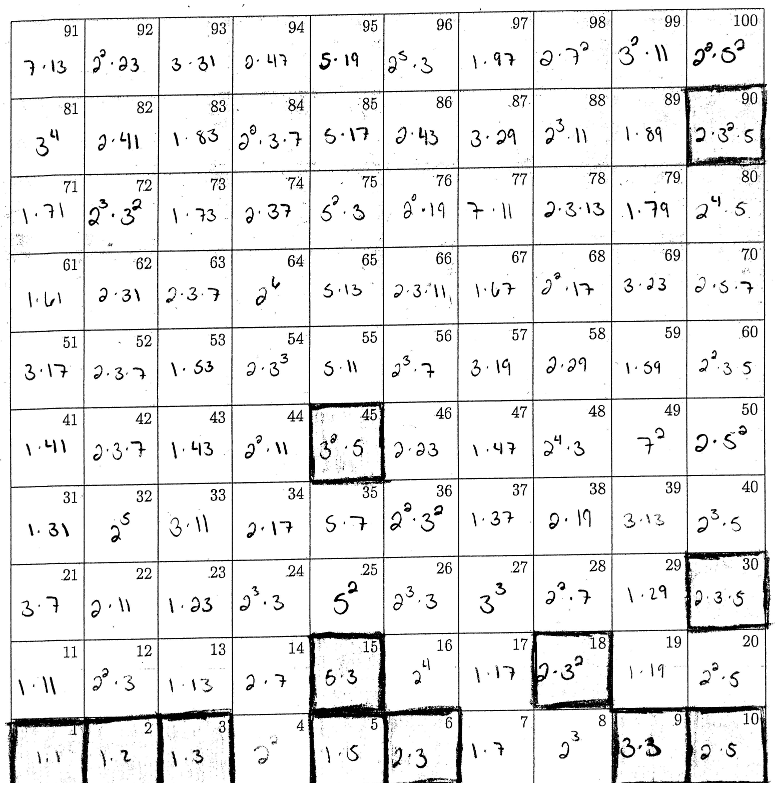

Figure 1: Example of participant work in Lesson 2

Table 1: *Description of Pre- and Post-Test Questions.*

Question	Prompt
1	Consider the number N = 3^2 x 5^4 x 11 x 17^3. Without calculating the value of N, determine whether each of the following is a factor of N. Justify each briefly. a) 5 b) 19 c) 15 d) 21 e) 75
2a	List all of the factors of 225. Show how you found all of them.
2b	List all of the factors of 5^2 x 7^2. Show how you found all of them.
3	What is the smallest positive integer that has the first ten counting numbers, 1 through 10, as its factors? Show or explain your work so that others can follow your logic. *Note: you may leave your answer in factored form.*

Findings

Analysis of the scoring data revealed that participants' abilities to solve number theory problems related to factors and prime factorization improved following the intervention described above. A paired sample *t*-test was conducted to compare participants' mean scores on the pre-test to their mean scores on the post-test. Results of the *t*-test indicated a significant difference between participants' pre-test (*M*=8.81, *SD*=4.40) and post-test scores (*M*=17.78, *SD*=4.97); *t*(68)=-13.88, $p < 0.05$. This suggests that participants' mean scores on the post-test were significantly greater than their mean scores on the pre-test. As such, the intervention supported participants' abilities to successfully solve problems related to factors and prime factorization.

Beyond this general result, the data provide information on how the intervention influenced participants' abilities to successfully identify prime factors, prime non-factors, composite factors, and composite non-factors. Prior research has shown that PSTs typically find it more challenging to identify non-factors than factors and composite factors than prime factors (Zazkis & Campbell, 1996a, 1996b). Table 2 shows mean scores, as a percent of available points on the scoring rubric, for Question 1a-1e on both pre- and post-tests. The table shows that participants improved in their ability to identify factors across all divisor types.

Table 2: *Mean scores for Question 1a-1e.*

Question	Divisor Type Given	Pre-Test	Post-Test
1a	Prime factor	62.3%	83.3%
1b	Prime non-factor	46.4%	81.2%
1c	Composite factor of form $p_1 \cdot p_2$	46.4%	78.3%
1d	Composite non-factor of form $p_1 \cdot p_2$	31.9%	70.3%
1e	Composite factor of form $p_1^2 \cdot p_2$	38.4%	75.4%

Interestingly, *differences* in PSTs' success rates on various types of problems appear to diminish following the intervention. Prior to the intervention, participants showed a marked difference in their ability to identify prime (62.3%) versus composite (46.4%) factors. Following the intervention, success rates in identifying prime (83.3%) and composite (78.3%) both increased while the difference between the two divisor types diminished. This result was maintained even when participants were faced with a more challenging composite factor as in question 1e. This reduction in performance differences following the intervention was also noted in participants' ability to identify factors versus non-factors. Prior to the intervention, participants were much more proficient at identifying prime factors (62.3%) than prime non-factors (46.4%) but were nearly equally proficient in these abilities following the intervention (83.3% and 81.2%, respectively). The result is similar in the case of composite numbers. Prior to the study, participants were more proficient at identifying composite factors (46.4%) than composite non-factors (31.9%). Following the intervention, success rates increased and differences diminished with a success rate of 78.3% in the case of a composite factor and 70.3% in the case of a composite non-factor.

Prior research shows that PSTs struggle to make use of a number's given prime factorization to identity its factors and instead revert to whole number conversion and long division. To assess the impact of the intervention on this area of concern, the study analyzed results obtained in questions 2a and 2b. In these questions, participants were asked to identify all of the factors of a particular number written in whole number (2a) and prime-factored (2b) forms (see Table 1). Participants were awarded credit in question 2a for finding all possible factors using any method. Participants were awarded credit in question 2b only if the participant made use of prime factorization in the construction of their response. Table 3 shows the mean scores, as a percent of available points on the scoring rubric, for both questions.

Table 3: *Mean scores for Question 2.*

Question	Pre-Test	Post-Test
2a	40.9%	72.5%
2b	25.5%	69.6%

The pre-test results replicate prior research by demonstrating that PSTs in the study exhibited greater difficulty in the identification of factors using prime-factored form (25.5%) as compared to whole number form (40.9%). Following the intervention, participants improved in their ability to find the factors of a number written in prime-factored (69.6%) and whole number form (72.5%) and, again, the difference in success rates in the two categories diminished substantially.

Conclusion

As the findings indicate, the number theory intervention is associated with a significant improvement in participants' abilities to solve problems related to factors and prime factorization. While the results of prior research appear to be validated by this study, the intervention shows promise in alleviating some of the challenges that PSTs have traditionally faced in learning these concepts. For instance, participants' ability to identify composite factors and non-factors improved substantially following the intervention, so much so that their success nearly equaled their success at finding prime factors and factors in general. One possible explanation for these improvements is that the intervention repeatedly asked participants to find composite factors and non-factors relying *only* on prime factorization. Increasing PSTs' exposure to prime factorization (Zazkis & Gadowsky, 2001) through activities incorporating visualization and exploration (i.e. Sinclair et al., 2004; Liljedahl et al., 2006) has shown promise in strengthening prospective teachers' understandings of factors. A larger, experimental study might confirm these results. The effect of teachers' different understandings of *factor* on students in their classrooms is also an area in need of greater investigation.

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A LEARNING TRAJECTORY FOR TRANSFORMATION-BASED REASONING IN GEOMETRY

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This project addresses the need to strengthen geometry instruction in the elementary grades by designing a conceptual framework around transformation-based reasoning upon which elementary geometry education may be constructed. We propose that engaging young students in the transformation-based concept of invariance in the contexts of both Euclidean geometry and topology can provide them with a stronger foundation for understanding the concepts of property, similarity, and congruence in more advanced Euclidean geometry, and also for recognizing the significance of invariance in other domains of mathematics. In this paper, we present our ongoing efforts to develop a learning trajectory for transformation-based reasoning.

Geometry in the United States is principally about identifying canonical shapes and matching those shapes to their given names (Battista & Clements, 1988; Clements, 2004). Learning is constrained to passive observation of static images of shapes on paper, which inevitably limits the engagement and understanding of geometry to holistic representations of shapes. Consequently, misconceptions arise. Students will argue that a triangle whose base is not horizontal is some other kind of triangle, or that a square and a regular diamond are distinct because they are oriented at different angles (Lehrer, Jenkins, & Osana, 1998, p. 149). They also develop narrow or prototypical images of geometric concepts such as right triangles, not realizing that right triangles may take on many forms (Vinner & Hershkowitz, 1980). The Common Core State Standards for Mathematics (CCSS-M) (CCSSO, 2010) impart much higher expectations for students' understanding of conceptual mathematics than what is typical of geometry instruction in U.S. classrooms.

Theoretical Framework

Research shows that young children possess a dynamic spatial sense of shape; they see them as malleable and often provide "morphing explanations" (Lehrer et al., 1998, p. 142) for shapes they identify as similar. To meet the expectations set forth in the CCSS-M, this dynamic sense may be leveraged within dynamic geometry environments that have been shown to formatively influence learners' understanding of geometry (Hölzl, 1996; Jones, 2001; Laborde, 2000). Such experiences hold the potential to nurture students' "geometrical eye," or "the power of seeing geometrical properties detach themselves from a figure" (Godfrey, 2010, p. 197), and provide a foundation for the formation of geometric conceptions, as their dynamic essence provides students with opportunities to connect a greater breadth of geometric concepts. Moreover, when the invariant aspect of properties is emphasized, the opportunities to make connections span other domains of mathematics, as well.

"Learning to 'see'" with a Godfrey's geometrical eye is what Johnston-Wilder and Mason (2005) regard as a central feature of the learning of geometry: engaging in active construction on the part of the learner in order "to become aware of relationships as properties that objects may or may not satisfy" (p. 4). Such an awareness requires a recognition of the distinction between non-defining attributes and defining attributes of shape. In this paper, we use the terms "attribute" for the former and "property" for the latter. In general, an *attribute* is a characteristic of shape that is variant upon transformation; a *property* is a characteristic that is invariant. In Euclidean geometry, any characteristic of shape that is invariant under rigid transformations is a property. In topology, however, these property-preserving transformations are invertible and continuous. Orientation is an attribute in Euclidean geometry; measure is an attribute in topology. Metric attributes such as length and angle measure are Euclidean properties; the non-metric attributes of connectedness and the openness of curves are topological properties.

Research Objective

The relationship between invariant properties and transformations is the fundamental feature of any geometry. Thus, we propose that students' engagement in what we refer to as "transformation-based reasoning," or TBR, can be a productive form of reasoning by which students develop a dynamic and connected sense of the relationships between geometric attributes, properties, and invariance. Further, we note that because Euclidean geometry is metric and topology is non-metric, the former is suited to a quantitative analysis whereas the latter is suited to a qualitative analysis. This distinction suggests that a transformation-based investigation of invariance in both Euclidean and topological spaces can support students' learning of these connected understandings, because the contrasts between transformation-based engagement in these spaces illuminate important distinctions between them (e.g., the cardinality of equivalence/congruence classes). Engagement in multiple geometries will not only alert students to the fact that there is more than one geometry, it will also support the learning of a more sophisticated representation of geometry and a more rewarding opportunity for engaging in geometric activity than what is typical of geometry instruction in elementary school.

Consequently, our investigation seeks to address the following research question: In the contexts of Euclidean geometry and topology, how does children's transformation-based reasoning develop over time?

Review of the Literature

The following review of the literature is organized by the connected concepts of attribute, property, and invariance, and the role of transformations in students' learning of these concepts.

The Attribute Concept. Students are expected to be able to classify objects by attributes as early as kindergarten (CCSSO, 2010). The identification and classification by attributes may begin as early as Pre-K when students' play involves the composition of a variety of shapes to form parts of a picture and the matching of shapes of the same size and orientation (Clements & Sarama, 2014). Subsequently, students learn to name basic shapes. Because they tend to be exclusively or primarily exposed to static and prototypical images of a triangle, students tend to misconceive of all triangles as equilateral. In addition, they identify triangles as only those three-sided polygons that contain a vertex at the "top" and a base at the "bottom," but not those polygons that have these attributes reversed (Confrey, 1992). In contrast, when students experience shapes dynamically, such misconceptions can be avoided or eliminated. By performing an increasingly sophisticated progression of rigid transformations and reflecting on their effects, students begin to discern characteristics of shape that are invariant upon transformation.

The Property Concept. Students are expected to distinguish between attributes and properties as early as first grade and to be able to build and draw shapes that possess particular properties (Clements & Sarama, 2014; CCSSO, 2010). Whether in a Euclidean or a topological space, students' dynamic manipulations of shapes enable them to begin to make these distinctions. They are able to do so, because transformations illuminate these distinctions, since only properties are retained in those manipulations. In a dynamic geometry environment, "dragging" transformations afford students an opportunity for "reasoning by continuity," a form of geometric reasoning that enacts Poncelet's Principle of Continuity: "The properties and relations of a geometrical system or figure, be they metric or descriptive, remain valid in all of the successive stages of transformation during a motion that preserves the definition properties of that figure or system" (Sinclair & Yurita, 2005, p. 5). Engaging students in this dynamic form of mathematical activity provides them with opportunities for exploring, experiencing, and making sense of properties of shape rather than having them transmitted directly.

Research has shown that preschool-age students are able to identify the effects of rigid geometric transformations on isolated figures (Moyer, 1978; Schultz & Austin, 1983; Xistouri & Pitta-Pantazi, 2011). As a consequence of using these transformations, they tend to avoid the popular misconception that all rectangles are similar because they all contain four right angles (Chazan, 1988). In the context of topology, children ages 6 and 7 also used mental transformations to identify a shape that is equivalent to a given one and to sort a collection of shapes into equivalence classes (Greenstein, 2014). Eventually, when students are introduced to *scaling* transformations, they can be supported to progress from "same shape, same size" and "same shape, different size" ways of understanding similarity and congruence (Confrey, 1992) to more formal distinctions by reflecting on the effects of these transformations. For example, reflection on the effects of scaling transformations that leave side length variant and angle measure invariant enables students to argue that two triangles are similar because their angles are congruent. Interestingly, this distinction between congruence and similarity is a Euclidean distinction. In topology, by contrast, size is a variant attribute, so equivalence classes contain shapes that are similar as well as those that are congruent. The capacity to distinguish between congruent and similar shapes in a Euclidean space, as well as the capacity to *not* make this distinction in a topological space, demonstrates a rather sophisticated understanding of property, especially for students in early elementary school.

The Invariance Concept. Elementary-age students are able to predict the effects of dynamic transformations by enacting transformations on mental representations of shape (Olive, et al., 2010; Panorkou et. al., 2014). In reflecting on the outcomes of those transformations, they are able to distinguish between variant and invariant attributes through abstraction (Piaget, 1970). Relative to the schemes that students assimilate as they then go on to use properties to construct Euclidean congruence classes or topological equivalence classes, this capacity parallels expectations put forth in the CCSS-M for Euclidean geometry that students understand that properties belonging to a category of shapes also belonging to all subcategories of that category (CCSSO, 2010). Greenstein's investigation of children's understanding of topology (2014) identified this capacity in one young child who gave names to equivalence classes that revealed her property-based distinctions and signified the structural character of those classes.

Not only is the concept of invariance fundamental to the distinction between concepts of attribute and property in geometry, but it also figures prominently across most other domains of mathematics. While Johnston-Wilder and Mason assert that "invariance in the midst of change is a central theme of mathematics, and particularly in geometry" (p. 2), Stroup (2005) goes so far as to propose that mathematics be viewed as a systematic study of forms of 'same-ness' or invariance: "Mathematical invariance is what allows us to see commonality in many situations that, on the surface, may appear to be very different" (p. 193). Thus, we claim that learners' systemic engagement in transformation-based ideas is not only productive for learning geometry, but also for opening new pathways into other domains.

Methods

Research on student learning in mathematics education over the past 20 years has yielded *learning trajectories* as an organizing framework for student conceptual growth (Clements & Sarama, 2014; Confrey, Maloney, Nguyen, Mojica, & Myers, 2009; Simon, 1995). Learning trajectories are "descriptions of the successively more sophisticated ways of thinking about a topic that can follow one another as children learn about and investigate a topic over a broad span of time" (National Research Council, 2007, p. 219). Thus, our goal is to design a learning trajectory of transformation-based reasoning. As a first step we have conducted a review of the relevant literature pertaining to the learning of concepts in Euclidean geometry and topology to develop the hypothetical trajectory of the development of Euclidean and topological concepts through transformation-based reasoning (TBR) that appears in Figure 1.

As the figure shows, the TBR trajectory spans learners' attribute-based understanding of shape, through property-based understanding, and culminating in the target understanding of the concept of invariance. The transformation-mediated engagement in these two geometries is the essence of TBR: distinguishing between variant attributes and invariant properties, and understanding that properties are determined by the nature of a group of transformations within a particular geometric space.

Now that we have developed an initial model of the learning trajectory, we will design and implement task-based clinical interviews (Piaget, 1976; Opper, 1977) to refine and validate the trajectory. Subsequent iterations of this process will involve the development of assessment items for clarifying the levels of the trajectory and think-aloud interviews to assess the quality of

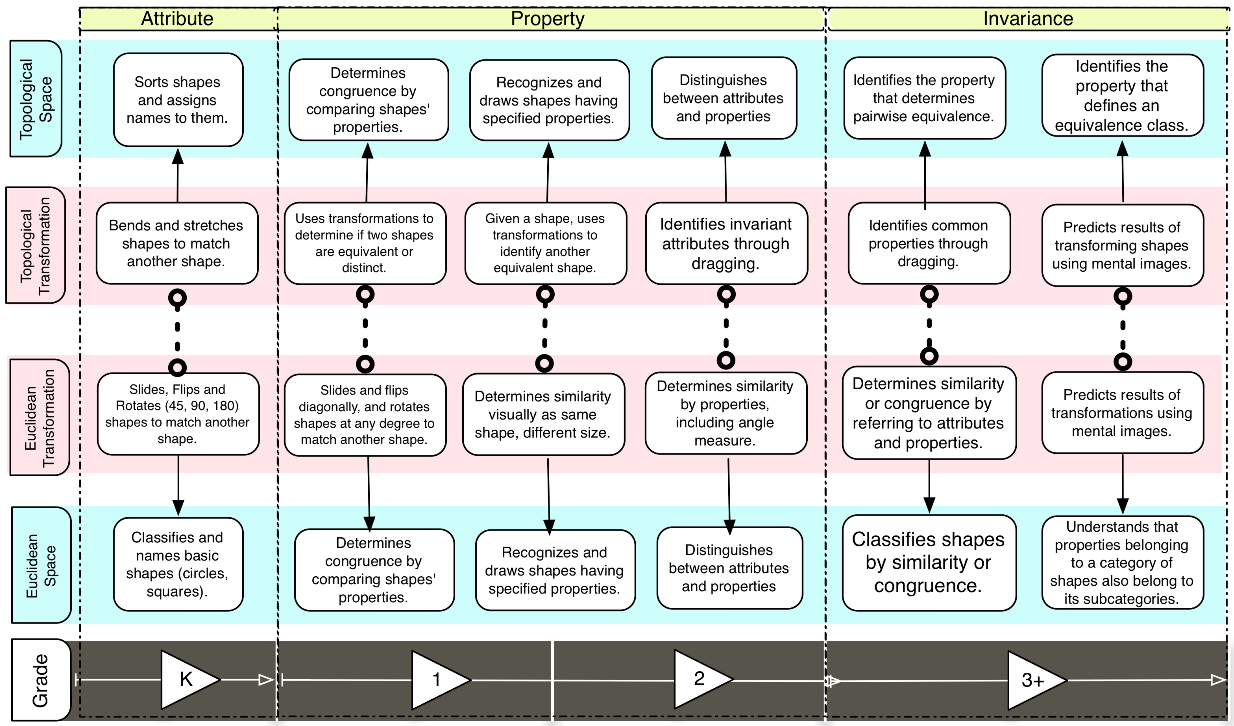

Figure 1. A learning trajectory for transformation-based reasoning

those items, eventually culminating in a viable model of the development of learners' transformation-based ideas in two geometries and a collection of tasks that have been seen to support this development.

Conclusion

Transformation-based reasoning is a conceptual framework that spans multiple geometries, connects a breadth of concepts within a geometry, and engages the multiple ways that children think or could think about geometry. Consequently, a validated learning trajectory of transformation-based reasoning should be beneficial to students whose geometric ideas are both metric and non-metric, to teachers who wish to realize the conceptual advantages of engaging their students in multiple geometries, and to curriculum developers who wish to develop materials to support the mutual development of these ideas.

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EMPOWERING OWNERSHIP OF PROOF WITH COMMUNAL PROOF-WRITING CRITERIA

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This study is embedded within an instructional sequence to help prospective secondary mathematics teachers develop communal criteria for proof. At four different universities, students were asked to complete proof-related tasks, evaluate fellow students' proofs in small groups, and then communally agree upon five criteria for evaluating a proof. Prospective teachers were empowered to negotiate communal criteria of proof with the instructor as facilitator. Results demonstrated similarities and differences in criteria across all four universities and developed a new metric comparing a prospective teacher's perception of proof to a classroom's perception of proof.

For many decades, proof has remained central to mathematics education as "the essence of mathematics lies in proofs" (Ross, 1998, p. 2). Recently, proof education at the secondary level has become accepted as vital to student understanding of mathematics (Harel & Sowder, 2007; Stylianides, 2007; Stylianou, Blanton, & Knuth; 2009; NCTM, 2014; Council of Chief State School Officers [CCSSO], 2010). However, the need for research in proof education for secondary teachers is still vividly apparent (Bieda, 2010). For example, Knuth (2002) and Yee (2014) found that pre-service and in-service secondary mathematics teachers are inclined to rely on textbooks, symbolic representations, and example-based evidence when deciding the validity of an argument. This is a significant problem for secondary students when their teachers promote mathematics as a set of rules from textbooks with guess and check as a main method to construct proofs (Bieda 2010; Knuth, 2002).

Traditionally, pre-service secondary mathematics teachers (PSMTs) primarily observe their instructor's polished and complete proofs when learning proof at the collegiate level (Stylianou et al., 2009). Within such learning environments, PSMTs often come to believe that their instructor has the sole authority to judge the validity of the students' proof productions (Harel & Sowder, 2007). To help PSMTs find the importance of learning proof, PSMTs need to be provided with proving opportunities within the classroom community. Specifically, Solomon (2006) emphasizes the need to not impose mathematical epistemologies of proof, but allow for situations where the classroom can create its own epistemologies of proof. In order for PSMTs to take ownership of their mathematical arguments, teacher educators must empower PSMTs within the classroom community to believe they have a voice in their epistemological constructions and creations of proof. To better understand PSMTs' criteria for proof, this study investigated what characteristics of proof PSMTs chose to be most valuable within their classroom community and how this communal rubric affected their understanding of proof. More specifically, this study was guided by the following two research questions: (Q1) What were the similarities and differences in criteria between classes? (Q2) How did the criteria affect self-evaluations versus class-evaluations of PSMTs' proof?

Theoretical Framework

To frame our study, we adapt the definition of proof proposed by Stylianides (2007). Stylianides' definition of proof is aligned with school mathematics and focuses on social negotiation and agreement of proof in the classroom community. Stylianides (2007) claims proof is a mathematical argument, a connected sequence of assertions for or against a mathematical claim, with the following characteristics:

> (1) It uses statements accepted by the classroom community (*set of accepted statements*) that are true and available without further justification;
> (2) It employs forms of reasoning (*modes of argumentation*) that are valid and known to, or within the conceptual reach of, the classroom community; and
> (3) It is communicated with forms of expression (*modes of argument representation*) that are appropriate and known to, or within the conceptual reach of, the classroom community (p. 291).

Stylianides thoughtfully created this definition clarifying its purpose and emphasizing school mathematics that balances mathematics as a discipline and a learning tool.

Recently more research has used student evaluations of proof to help deconstruct empirical proof schemes (Harel and Sowder, 2007). Ko and Knuth (2013) found that many students struggled with what defined an argument because students would accept a proof and a counterexample to that proof simultaneously. Bleiler, Thompson, and Krajčevski (2013) had 34 preservice secondary mathematics teachers evaluate preconstructed high-school arguments to help transition students towards a deductive proof scheme by helping students articulate and identify errors in high-school students' work.

Our study builds upon prior studies by focusing on empowering the classroom community to fully create, categorize, and define proof. Our study is also designed to determine how students validate proof, but will focus on what aspects of proof they value. Knuth (2002) and Bieda (2010) both found that teacher beliefs significantly influence their practice as teachers. To determine those beliefs, our study chose to have students evaluate each other's proofs so that hegemonic influence of the instructor's preconstructed arguments did not influence students' understanding of proof. Fundamentally, prior studies still evaluated students by the researchers' preconstructed arguments/evaluations. This study's design contributes to the field's understanding of proof education because it (1) uses the students' arguments in the classroom activity instead of preconstructed arguments (2) allows the classroom community to fully define proof criteria with the student lens instead of an external rubric of the instructor or researcher.

Method

This study was conducted by four researchers at four separate courses across four institutions who were the instructors and authors of this paper. Two of the courses (Authors 1 and 4) were mathematics content courses, and the other two (Authors 2 and 3) were secondary mathematics methods courses. There were a total of 55 participants, including undergraduate and graduate students who are mathematics or secondary mathematics education majors. As those with degrees in mathematics could decide to become teachers in the future, it is possible that all the participants in this study are PSMTs.

Design of the Instructional Sequence

The instructional design was organized around the Sticky Gum Problem (Figure 1) into a pre-class activity, an in-class activity, and a post-class activity (Table 1).

The Sticky Gum Problem

Ms. Hernandez came across a gumball machine one day when she was out with her twins. Of course, the twins each wanted a gumball. What's more, they insisted on being given gumballs of the same color. The gumballs were a penny each, and there would be no way to tell which color would come out next. Ms. Hernandez decides that she will keep putting in pennies until she gets two gumballs that are the same color. She can see that there are only red and white gumballs in the machine.

1) Why is three cents the most she will have to spend to satisfy her twins?
2) The next day, Ms. Hernandez passes a gumball machine with red, white, and blue gumballs. How could Ms. Hernandez satisfy her twins with their need for the same color this time? That is, what is the most Ms. Hernandez might have to spend that day?
3) Here comes Mr. Hodges with his triplets past the gumball machine in question 2. Of course, all three of his children want to have the same color gumball. What is the most he might have to spend?
4) Generalize this problem as much as you can. Vary the number of colors. What about different size families? Prove your generalization to show that it always works for any number of children and any number of gumball colors.

Figure 1. The Sticky Gum Problem originally created by Fendel et. al. (1996).

Table 1. *Instructional Sequence for Sticky Gum Problem (SGP)*

Pre-Class Activity	• Students attempt to solve the SGP and submit their response to the instructor. • The instructor chooses five distinct student arguments and makes copies for each student.
In-Class Activity	• The instructor breaks the class into small groups of 3-4 students. The students read the five selected, distinct arguments by their fellow students. • Small groups decide whether each of the five arguments are valid proofs by creating a list of five criteria that define proof. • The small groups all share in a class discussion about what defines proof and a list of five communal criteria are agreed upon by each class.
Post-Class Activity	• Students are to rate their initial argument according to these five communal criteria. • Students are to self-evaluate their argument within each criteria on a scale of 0 to 5. • Students are to rewrite their argument for the SGP to satisfy the five communal criteria.

We chose the Sticky Gum Problem (SGP) for three reasons: (1) It involves multiple variables; (2) It is appropriate for middle-school through college students; and (3) It provides opportunities for students to create specific examples, look for patterns, make generalizations, and construct arguments via probability and algebraic reasoning.

As stated in Table 1, students submitted their solutions to the SGP, evaluated five instructor-selected student proofs of the SGP, created five class-communal criteria from these evaluations, and then used them to evaluate their original argument (on a 0-5 scale) and rewrite a new SGP argument. During the in-class activity, small groups discussed and agreed upon which arguments were valid proofs. This scoring system provoked important conversation about what defines a proof. The classes were then instructed to articulate five characteristics of proof as criteria for determining arguments to be proofs. After each small group created a list of five characteristics, all groups shared their criteria with the class. The whole class then discussed and agreed upon what should be appropriate criteria for proof. Often groups negotiated the meaning of words such as *clarity, evidence, and concise*. These negotiations led to valuable conversations that allowed students (many for the first time) to consider what was significant to their understanding of proof. Fundamentally, this encouraged communal ownership for definitions of mathematical proof.

Research Design and Data Analysis

Using the naturalistic paradigm with the instructional sequence, this mixed methods study answers the previously-stated research questions. Regarding data analysis, each instructor shared their student-constructed criteria for the class with all other researchers after completing the in-class activity. All researchers agreed on which criteria were coded similarly and which were different. All of the authors stayed true to their students' responses and wording was not changed in how students defined their criteria for their class. Next, an ANOVA between the five featured students from each class compared the class's evaluation of their proof to their own evaluation of their proof. With so much data and such limited space, we will only be reporting on the communal criteria categories and the student/class evaluations.

Results

Q1: What were the similarities and differences in criteria between classes?

Table 2 illustrates each classroom's agreed upon categorization and five criteria.

Table 2. *Communal Criteria by Researcher's Class*

	# of Agreed Researchers	Researcher1, University 1	Researcher2, University 2	Researcher3, University 3	Researcher 4, University 4
LOGICAL STRUCTURE	4	Organization-Structure of Argument.	Logical progression / no math errors.	Proof needs to follow a logical order.	Chain of evidence.
GENERALIZABLE	4	Convincing-Exhaustive, not ambiguous, audience appropriate.	True for all cases	Proof should be always true for any cases and it should not be verified by specific examples.	Generalization.
CLEAR AND CONCISE	3	Clarity-Concise, "Math Language" and "Grammar"		Proof needs to include clear explanations that are concise.	Simplified/Concise.
STATEMENT AND CONDITIONS	3		Clearly stated conjecture.	Proof includes clear statements of what you are trying to prove.	Clearly identifying parameters/constraints.
DEFINITIONS	2	Definitions	Clearly define your domain, definition and assumptions.		
VALIDITY	2	Correctness-Supporting evidence, overall structure.			Valid/true/correct.
CONVINCING	1			Argument must be clear for audiences to follow based on their community.	
STATE CONCLUSION	1		State a conclusion that follows from the argument.		

As shown in Table 2, all classes agreed with a need for logical structure, lending credence to the use Stylianides' (2007) definition for proof because students are addressing the need for *modes of argumentation.* Similarly, the majority of the classes (3/4) valued clear and concise explanations as their *modes of argument representation.* One can also observe the majority of the classes (3/4) chose the criteria for statements and conditions, following Stylianides' *set of accepted statements.*

When looking at criteria that half or fewer of the classes agreed upon, we see Author2's students emphasizing the specific format for writing a proof, such as "State a conclusion that follows from the argument". This class' understandings of proof may have been influenced by their previous instructor or textbooks. It is important to notice that no category aligned with the need for empirical proof schemes (Harel & Sowder, 2007). Strict instructional design restricted all researchers from manipulating class criteria, and yet there are no references to inductive reasoning as valuable criteria for evaluating a proof. In fact, all the classrooms agreed upon generalizability, which is an example of how students chose *not* to accept empirical data (e.g. "True for all cases" and "Exhaustive").

Q2: How did the criteria affect self-assessment vs class assessment of students' proof?

In the post-activity, students were asked to rate their original argument using the classroom-developed criteria for proof. Under each criterion, students gave their original argument a score from 0-5. Because the criteria differed from class to class, we could not compare all classes and all criteria. However, we did collect data on the five selected students' proofs evaluated by each class during the in-class activity and self-evaluated at home by the individual student.

During each class, students rated the five chosen student's work as either a proof or not a proof (0 or 1). We also had 17 complete individual self-assessments in the post-activity. As the criteria for the self-assessment varied from class to class, we averaged the criteria scores in the self-assessment to find students' average self-rating with the class criteria. For example, consider Susan's initial argument solving the SGP (Figure 2).

Let c be the number of colors of gumballs in the gumball machine. Let k be the number of kids that want the same color gumball in a family. We need to show that g(c,k)=[(c)(k-1)]+1 represents the maximum number of pennies a parent would have to spend to satisfy their kids. If we multiply c by (k-1) then either

1. Each color came out (k-1) times and (k-1) kids have all c colors.
2. Or at least one color came out at least k times, because another color came out less than (k-1) times.

If **2.** happens then all of the kids have the same color gumball and they are happy. If **1.** happens then one more kid needs a matching gumball. In this case the next gumball will satisfy the k^{th} kid because the first (k-1) kids already have all c colors. Thus the most a parent would have to spend is g(c,k)=[(c)(k-1)]+1 colors.

Figure 2: Susan's initial argument to solving the Sticky Gum Problem.

Susan gave herself an 88% self-rating, but only 75% of the class considered her justification a proof. This surprising result led to further quantitative analysis on the variance between the individual's assessment using the communal criteria and the class' assessment using communal criteria.

Students rated their proofs (M=63.88%, SD=22.70%) higher than the class rated their proofs (M=43.43%, SD=27.41%). To confirm this variance, a one-way analysis of variance (ANOVA) was used with all 34 assessments (17 self-assessments, 17 class-assessments) to determine if the selected students proofs class-score aligned with a student's individual score. To make sure the F-Test for the ANOVA was appropriate, normality, independence, and homoscedasticity was tested. The Levene Statistic ($F(33)$=2.302, p=0.139) showed that the homogeneity of variance was not violated, supporting the validity of the ANOVA. With an alpha level of five percent, there was significant variance between self-related scores and class-related scores ($F(33)$=5.62, p=0.024). While students all used the class-developed criteria for proof to evaluate their original argument, it did not mean that they were critical of their work or possessed sufficient mathematical knowledge needed to evaluate mathematical arguments. This is concerning as we would expect that self-evaluation of proof to be reflective of the class' evaluation of the same proof. By comparing the variance of an individual student's argument to that of the individual's evaluation of their own argument, this study has created a new mathematics education metric that can determine if a classroom's perception of proof aligns with the individual's perception of proof.

Discussion

Research has documented that PSMTs experience difficulty constructing and evaluating mathematical arguments (Bleiler et al., 2013; Harel & Sowder, 2007). One of the primary challenges PSMTs face in developing an understanding of proof is being aware of the external conviction proof scheme (Harel & Sowder, 2007); PSMTs typically follow their instructor's or textbook's proof formats and apply these strategies to similar problems. To provide opportunities for PSMTs to learn proof as communal, negotiated, and sense-making process, this study implemented an instructional sequence tailored to elicit communal proof criteria using students' arguments with the SGP. Using a mixed methods approach, we found that there were similarities across all four universities where PSMTs were the majority of students. Specifically, all classrooms agreed that logical structure and generalizability were important criteria for proofs, while three out of the four agreed proofs should be clear and concise, and state conditions/necessary statements.

The class-developed criteria for proof also offered insight into how the individual students perceived their arguments versus how the class perceived individuals' proofs. An ANOVA illustrated that how students evaluated their own proof was significantly higher than how the class as a whole evaluated their proof. Why self-assessment was higher than class assessment was not built directly into this study as it wasn't anticipated, but raises valuable questions for future studies. Ideally, we would want the classroom evaluation and individual evaluation to align. Thus we have a new metric for aligning a proof not with the instructor's rubric, but with the class criteria. This has enormous potential for comparing proof material across mathematical topic as well as classrooms offering a means to compare proof evaluation in larger quantitative studies.

This study encouraged discourse about what counts as proof by having PSMTs assess and create criteria for what should count as proof. By empowering PSMTs to critically evaluate each other's proofs, and not a set of pre-constructed arguments, we found valuable results in what criteria they chose, and how they chose to use those criteria to evaluate proof. Not only do these communal criteria empower future teachers to take ownership of proof in their classroom, but also builds on prior proof research to move the field towards a more classroom-centered use of proof education.

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UNDERSTANDING INTEGER AND BINOMIAL MULTIPLICATION

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The researchers of this study conduct a series of design experiments with prospective middle grades mathematics teachers that is focused on multiplication and division of integers and polynomials. Adaptations in the process occur as to how questions are asked as well as how tasks are scaffolded. Two consistent findings emerge. First, while participants can perform the algorithm, they are limited in their understandings of why a negative times a negative results a positive integer. Second, participants struggle with binomials and connecting them to real-life applications. Implications for future research are discussed.

Our research examines the effectiveness of teaching algebraic concepts involving the NCTM (2000) *Process Standards*, including representation, connections and communication. We believe that by providing process standards, NCTM is fostering the notion of moving away from "teaching-as-telling," a common practice in mathematics teaching (Doll, 2005; Lobato et al., 2005; Smitherman-Pratt, 2006). In the research we conduct, we utilize these standards in particular ways. For us, representation as a process allows for "imaging" in order to conceptualize mathematics (Wheatley, 1998; Richardson, Pratt & Kurtts, 2010). Second, communication involves the articulation of ideas in a cohesive manner that conveys ideas. Because the metacognitive act of communication can be a difficult skill to develop, in our research we scaffold the tasks to gradually transition from numerical sense to algebraic abstractions. Third, connections of numbers and algebraic representations are more explicitly demonstrated so that conceptual understandings of abstract ideas may be well established.

Theoretical Context

As mathematics educators, we are interested in teaching mathematics for understanding. As mathematics education researchers, our primary focus is examining if the pedagogical practices being used accomplish this. We work with prospective middle grades mathematics teachers, with the goal of assisting them to distinguish the difference between exposure and understanding of mathematical knowledge (Asquith, 2007, p. 268). Our research is framed by the general notion of mathematical knowledge for teaching, which is described by Deborah Ball and colleagues (Ball, 2003; Ball, Thames & Phelps, 2008) as a relationship between pedagogical content knowledge and mathematical knowledge. The methodological framework employed in this study is design experiment (Cobb et al., 2003), a method of collecting data in order to explore how to "to develop theories about both the process of learning and the means designed to support that learning" (Gravemeijer & Cobb, 2006, p. 18). Design experiment includes a three-phase cycle:

1. Create a local instruction theory based on research and experience;
2. Continuously test the theory during the experiment; and,
3. Complete a retrospective analysis after completing the experiment to refine the local instruction theory.

We follow this cycle over three distinct experiments, in 2009, 2008 and 2012, modifying both within an experiment and before subsequent experiments. Design experiment allows for emergent adaptation while in the midst of the study, which we employ during our data collection.

Research Design

Participants in the studies are prospective middle grades mathematics teachers who are enrolled in a middle grades mathematics methods course. The participants from 2008 and 2009 are located at a university in the south, and in 2012 at a different university located in the southwest. Each experiment includes data collection every other week over a total of seven weeks (three to four days total per experiment). Participants work in small groups and also participate in whole-class discussions. Data collection includes audio, video and written artifacts.

The research questions, maintained across all three design experiments, are as follows:

1. How does a prospective middle grades mathematics teacher conceptualize multiplication of integers and polynomials?
2. Does that knowledge change when engaged in tasks that include processing through representation, communication and connections?

The pre- and post-test assessments are word problems generated by each participant (Ma, 1999). Analysis of the word problems follow a rubric created by the researchers to assess change in understandings, as demonstrated in the word problems.

Research Implementation

During each experiment, participants use base-10 blocks & *Algeblocks*. Accompanying the *Algeblocks* are special mats for the blocks. The basic mat is for collecting like terms, and the quadrant mat is for multiplying and dividing/factoring integers and expression. Participants are given process sheets (see *Figure 1*) to record their blocks to coincide with each given task. Our intention in incorporating both the manipulatives and the process sheets is to assist in transitioning from the concrete to the abstract, allowing the prospective teachers to view manipualtives as more than just fun (Moyer, 2002). While there has been research on the use of algebra tiles (e.g.,; Boston, 2013; Caglayan, 2012; Kilhamn, 2011), there is no research to date on *Algeblocks* (which are similar in focusing on an area model but structurally different from algebra tiles and also includes mats). A more detailed comparison is currently in draft.

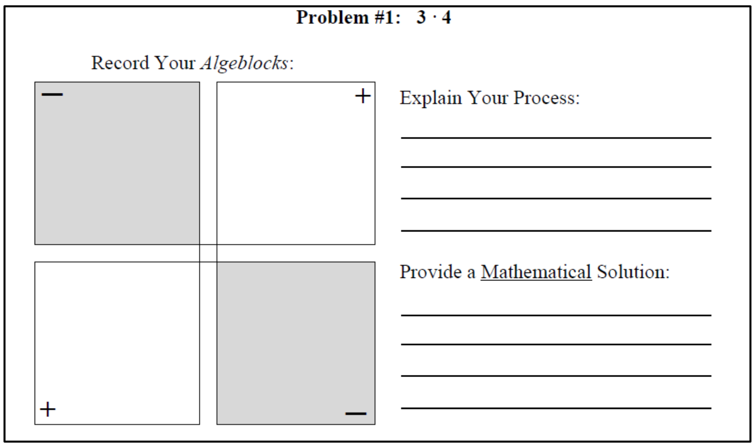

Figure 1. The first process sheet used in Day 1.

The first experiment follows this schedule of tasks:

- Day 0: Collecting like terms using the basic mat
- Day 1: Multiplying and dividing/factoring using the quadrant mat
- Day 2: Multiplying and dividing/factoring using the quadrant mat

After the first day of the data collection in 2008, the researchers identify that the process sheets would not assist participants in their assessment of constructing a word problem at the conclusion of the experiment. Therefore, a change is made from Day 1 to Day 2. The new process sheet, which is used for all subsequent tasks, includes a prompt both before and after the recording of the models. Beforehand, to elicit opportunities to communicate their current understandings, we ask, “Explain, in words, what you think this problem is asking.” Then after the task is represented with the models, the informal process, and a formalized mathematical algorithm, we ask, “Provide a ‘real-world’ context for this expression.” Our purpose for the initial prompt is to listen to their conceptions of the mathematics. For the final part, the attempts to apply the mathematics generates opportunities for communicating what is now understood and also connecting the mathematics to life. Our intention is to build toward the post-assessment task of a word problem by asking for scenarios along the way.

Another significant change is made following the experiment in 2008. The scaffolding of the tasks for multiplication and division is made, and the new arrangement is used in both 2009 and 2012:

- Day 0: Collecting like terms using the basic mat (same as before)
- Day 1: Multiplying and dividing/factoring with the set of whole numbers (stay in Quadrant I)
- Day 2: Multiplying with the set of integers
- Day 3: Dividing/Factoring with the set of integers

The primary reason for this change is the recognition that the cognitive load of understanding multiplication and division using an area model and also understanding the multiplication of two negative integers is too demanding to adequately address simultaneously. Thus, the first day is changed to focus only on whole number and binomials with variables as whole numbers then to dividing/factoring whole numbers and trinomials. The next day focuses on multiplication across all four quadrants (expanding from whole numbers to integers). This allows for a substantial amount of time for participants to represent and communicate why a negative integer times a negative integer results in a positive. The final day focuses on dividing and factoring integers and trinomials with variables as integers. During this day the idea of adding in zero pairs to construct rectangles is explored and connects back to the collecting like terms that the participants investigated prior to data collection.

After the experiment in 2012, the researchers identify that the prompts for the assessment needs to be changed. All three experiments require the participant to individually "Write a word problem that requires the multiplication of two binomials;" and, "Write a word problem that requires the multiplication of two integers." The suggested change pertains to the second word problem, "Write a word problem that requires the multiplication of two <u>negative</u> integers." Because the original prompt does not specifically state negative integers, the participants may or may not include two negative integers in their word problem, which is of primary importance in this research. In future studies, these prompts will be used.

Research Analysis Tool

In order to answer the second research question as provided above, the researchers use the following rubrics to analyze the word problems. (*See Table 1*).

Table 1. *Rubric for Binomial Multiplication and Negative Integer Multiplication*

The application problem created by the participant:	
– uses only binomials.	– uses the set of integers.
– requires the use of multiplication.	– uses negative integers.
– requires the use of two binomials.	– requires the use of multiplication.
– is not too complex for middle school students.	– is appropriate for the use of negative integers.

Research Findings

In this paper, we will limit our analysis to 2012 participants (7 in all). *Tables 2 & 3* provide a summary of the 2012 of the pre-assessment and post-assessment scores. (Pseudonyms* have been used for all participants.)

Table 2. *Multiplication of Two Binomials Gains*

Participant*	Pre-Assessment	Post-Assessment	Gains
Anna	0	3	3
Mrs. D	2	2	0
Frank	0	4	4
Logan	1	3	2
Nicole	2	4	2
Elizabeth	3	4	1
Julie	0	3	3

Table 3. *Multiplication of Integer Multiplication Gains*

Participant*	Pre-Assessment	Post-Assessment	Gains
Anna	0	1	1
Mrs. D	1	3	2
Frank	4	2	-2
Logan	4	4	0
Nicole	2	4	2
Elizabeth	1	3	2
Julie	4	4	0

Almost all of the participants increased their abilities to apply algebraic expressions, thereby showing that the use of representation, communication, and connections do help.

One unique example is the results from Frank. During Day 2 (multiplication in all four quadrants), the participants are asked to model the problem of (-1)(-4). After showing their product in Quadrant III, they work to create a real-life example of this.

Elizabeth: *"That concept of negative sets is difficult. How do you go in the negative direction?"*

Frank: *"Go back to money. You owe me $4 and I remove the debt. So how much money am I essentially giving you? $4."* (16:55 -17:30)

However, when prompted again, Frank responds by saying that he is struggling with a scenario because, *"I'm still stuck in my brain about how I learned it... "*(21:09-21:32). As evidenced in his net loss in constructing a word problem at the conclusion of the experiment, Frank struggles to make sense of a negative times a negative integer. Stein et al. (2000) discuss how those who are first taught the procedure then shown manipulatives often have more barriers in being able to understand concepts. In contrast, Elizabeth seems to work through her initial inability to compose a word problem by the conclusion of the experiment, as shown in her net gain.

Discussion

Two major aspects of the experiments are noteworthy. First, the initial reason for the study is to decipher how prospective mathematics teachers interpret the meaning of the multiplication of integers. The quadrant mat assists in representing *why* a negative integer times a negative integer results in a positive integer, which allows for an opportunity to develop conceptual understandings. Further, by using the blocks to explore both multiplication and factoring, participants' conceptual understandings are developed with respect to the Distributive Property of Multiplication over Addition as both a (b + c) = ab + ac *AND* ab + ac = a (b + c). This leads to an understanding of how factoring connects to the Distributive Property and also generates stronger connections between multiplication and division/factoring. We believe that the incorporation of the tasks and the conversations around them encourages the participants to "unpack one's thinking through the use of representations as explanatory tools" (Charalambous, Hill, & Ball, 2011).

A second aspect is what emerges in the first experiment and has become a focus for analysis in subsequent experiments, namely an apparent lack of conceptual understanding of a binomial. Our current data suggests that the use of *Algeblocks* assists prospective mathematics teachers to better understand mathematical concepts involving binomials. Also, the *Algeblocks* represent why seemingly procedural math processes such as distribution and factoring actually work, thus distinguishing a difference between doing mathematics and understanding mathematics.

Conclusion

One of the limitations of this data is that the participants in each experiment are students enrolled in the only section of this course. No comparison group exists within the context and time for each group. A second limitation is that the instructor of record is also one of the researchers.

There are several recommendations for future studies. First, as stated above, a change of the integer word problem for pre- and post-assessments will more clearly measure a participant's understanding of the multiplication of two negative integers. Next, we recommend giving both pre-assessments at the start of Day 1 instead of the staggering them. Lastly, in order to collect more qualitative data of individual participants' understandings, we recommend conducting structured interviews. This will supplement the data collected during the experiments in groups and whole-class discussions.

In this series of experiments, the researchers have gained insight with respect to how middle grades pre-service teachers struggle to apply the concept of the multiplication of negative integers and of two binomials. Procedurally, they could solve these; conceptually, they have difficulty explaining why the procedures are true. By providing the blocks and the quadrant mat, the participants can visualize multiplication as an area model and also make connections between the coordinate plane and operations of integers and polynomials. This is significant because we believe it connects well with functions, a key idea in algebra.

Overall, the results from this experiment show that participants can improve their abilities to compose word problems that involve the multiplication of two integers and also binomials. Word problems are one way to examine the ability to communicate conceptual understandings. By using the base-10 blocks and *Algeblocks*, participants can create mental images of the mathematical operations, thereby being able to represent the processes both geometrically and algebraically. Engaging in these tasks allowed participants to actively construct their own knowledge through representation, connections and communication. We believe this will impact their future instruction because teachers need to be ready to explain why, show how, and give alternate equivalent representations. They need to be able to predict what common mistakes will be made and come up with appropriate definitions of mathematical terms that apply to the students at their current stage in math (Ball, 2003).

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INVESTIGATIONS INTO MATHEMATICS TEACHERS' PROPOSITIONAL LOGIC CAPABILITIES

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This paper describes the reasoning processes of prospective and inservice secondary mathematics teachers on a conditional statement involving mathematical logic. Although this question has been studied for over 50 years in general settings, our research deals with participants with presumed course preparation in collegiate mathematics to successfully answer the question. The research is tied to content knowledge for teaching, psychological research on deductive reasoning, and the Common Core Standards for Mathematical Practice. In general, participants successfully saw one line of reasoning, but over 50% did not successfully use a second line of reasoning and obtain a correct solution.

This paper discusses the responses of prospective and inservice mathematics teachers on a conditional reasoning task that is, "the most intensively researched single-problem in the history of the psychology of reasoning" (Evans, Newstead, & Byrne, 1993, p.99). The task was presented with a picture of four cards with the statement that each card contains a circle or star on one side and either a triangle or square on the other side. Participants were asked to chose which card(s) must be turned over to verify the statement: "Every card with a star on it has a triangle on it." Studying the responses given by the participants gives insights into the content knowledge gained from their collegiate mathematics preparation related to conditional statements, and addresses aspects of the research question: How do mathematics teachers who are being prepared in the era of the Common Core State Standards for Mathematics respond to problems involving mathematical logic?

Background and Literature

The Mathematical Education of Teachers II (MET II) (Conference Board of the Mathematical Sciences, 2012) is the one singular document that best describes desired content preparation for secondary mathematics teachers including detailed descriptions of courses in which prospective teachers should engage. MET II emphasizes that "A primary goal of a mathematics major program is the development of mathematical reasoning skills" (p. 55), and offers the following advice on how to accomplish this.

> In order to be able to recognize, foster, and correct their students' efforts at mathematical reasoning and proof, prospective high school teachers should analyze

> and construct proofs themselves, from simple derivations to proofs of major theorems. Also, they need to see how reasoning and proof occur in high school mathematics outside of their traditional home in axiomatic Euclidean geometry…Prospective teachers can gain experience with reasoning and proof in a number of different courses, including a dedicated introduction to proofs course for mathematics majors, Linear Algebra, Abstract Algebra, Geometry, or a course on high school mathematics from an advanced standpoint. (pp. 58-59)

In briefly discussing how prospective teachers answered questions from classroom scenarios, the authors of the MET II report cited the work of Shulman (1986) and Ball (1990a) to indicate that even mathematics majors did not always answer questions from pre-college mathematics in a satisfactory way. Ball (1990b) found that although prospective secondary mathematics education candidates believed they understood mathematics and were confident in their own ability to do so, they were no more successful than prospective elementary teachers when it came to providing conceptual explanations for mathematics concepts.

There are several psychological lines of research in which the question we posed to the participants in our study can be situated. Ennis (1975) discusses it in the context of children's ability to handle propositional logic including what children can do as well as the importance of the context and structure of the question. Citing numerous studies, Ennis writes, "What is it that children cannot do that adolescents can do? Both have the ability to reason in accord with at least some of the principles of propositional logic, and both have considerable trouble with logical fallacies" (p. 24). Stylianides & Stylianides (2008) discuss work related to the question we posed to the participants within the context of proof and deductive reasoning connections to mathematics education. This question is attributed to Wason (1966) and is considered a classic with respect to the psychology of deductive reasoning. When participants are asked which card(s) must be turned over to verify the statement *every card with a star on it also has a triangle on* it they are investigating how to determine the truth value of mathematical implication (P --> Q), where P corresponds to *If a card has a star on it* and Q corresponds to *then it has a triangle on it*. This aligns with reasoning in mathematics and is discussed and used in any upper division mathematics course where justification and proof are studied or used. There has been considerable interest in this question as well as to whether or not the social context of the question matters (e.g., Politzer, 2004a, 2004b; Cosmides & Tooby, 1992; Johnson-Laird, 1983;

Johnson-Laird & Byrne, 1991; Cheng, Holyoak, Nisbett & Oliver, 1986). Stylianides & Stylianides (2008) discuss two theories of deductive reasoning, mental models and pragmatic reasoning schema theory, important to proof. In mental models theory a person creates a mental representation with a structure similar to the involved situation while pragmatic reasoning schema theory involves generalized abstracted rules. Due to the relationship of proof, both of these apply to the work described because from a mathematical point of view there is a structure to this problem, and in using the structure and appropriate reasoning strategies, a logical reason can be used to answer the question. Given the recommendations of the MET II report, it is our contention that prospective or inservice secondary mathematics teachers are expected to have the knowledge necessary to answer the question posed within either one of these constructs of deductive reasoning.

Additionally, we have specifically identified two Standards for Mathematical Practice (SMP) in the CCSSM related to this research investigation. Even though the SMP are for K – 12 grade students, teachers are charged with setting the classroom culture for students. This research gives some indication how prospective and inservice teachers engage in these practices while solving a task involving mathematical logic. These SMP are,

> **SMP 2** – Reason abstractly and quantitatively:
>
> They [students] bring two complementary abilities to bear on problems involving quantitative relationships: the ability to decontextualize—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, …—and the ability to contextualize, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand. (NGA Center and CCSSO, 2010, p. 6).
>
> **SMP 3** – Construct viable arguments and critique the reasoning of others:
>
> Mathematically proficient students understand and use stated assumptions…. They make conjectures and build a logical progression of statements to explore the truth of their conjectures….are able to analyze situations by breaking them into cases….justify their conclusions, communicate them to others…. (NGA Center and CCSSO, 2010, p. 6 –7).

Methods and Findings

Three groups of prospective and inservice mathematics teachers responded to the question. In one group (n = 10), students were enrolled in a required upper division Abstract Algebra course required for a degree in mathematics or mathematics education at a Midwestern United States institution of higher education. They answered the question during the last two weeks of a semester. The students in the second (n = 18) and third (n = 20) group were enrolled in graduate-level (Masters) secondary mathematics methods courses at an institution of higher education in the Western United States. Based on the courses in which they were enrolled it is assumed participants had sufficient mathematical background to answer the question correctly.

The question and format for responding was presented as shown in figure 1.

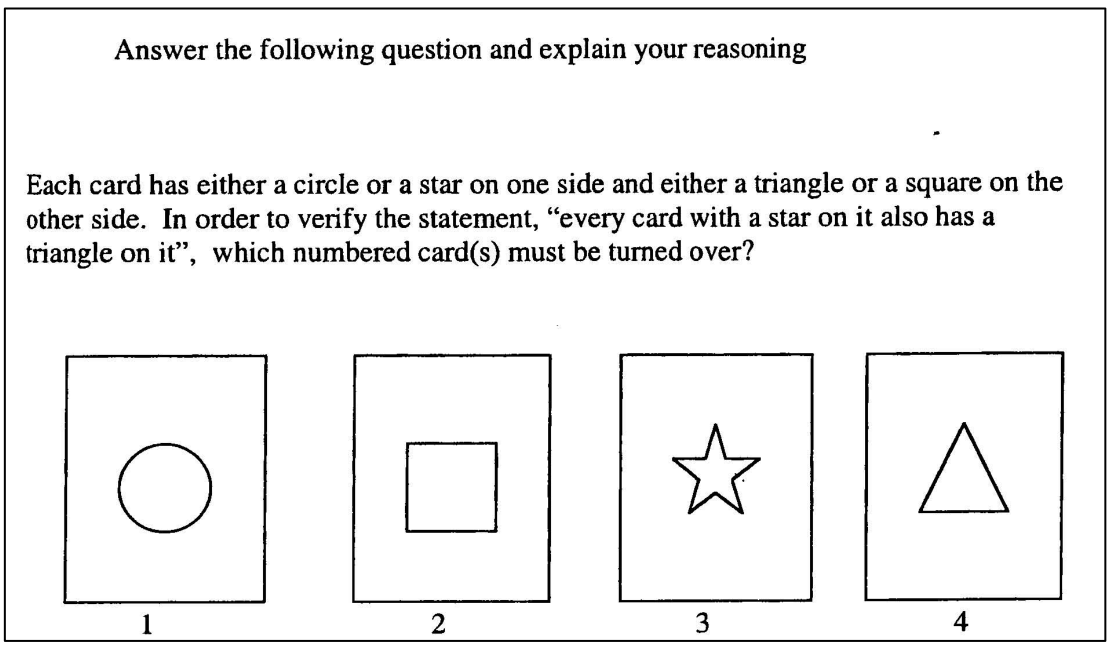

Figure 1. Question given to participants

Data for the three groups are combined for this paper (n = 48). Participant responses were first coded by correctness of the answer provided, and then by reasoning for only those answers that were correct. These data are shown in Table 1, with 16 responses (33.3%) correctly and 13 (27.1%) correct with justifiable reasoning.

Table 1. *Number (%) of correct answers and correct answers with correct reasoning, n = 48*

Correct Answer (Cards 2 and 3)	Correct Answer and Correct Reasoning
16 (33.3%)	13 (27.1%)

According to Stylianides & Stylianides (2008, p 121), "The correct selection among adults on this (standard) form of the task is usually about 10%." Hence, the data shown indicates the participants answered correctly more than the general population. Seven of the responses with a correct answer and explanation indicated why Cards 2 and 3 must be selected, as well as why Cards 1 and 4 do not, thus examining conditions of each case. One such example is

> *Only cards 2 and 3 could have stars with something other than a triangle on the other side. If card 2 has a star on the other side it disproves the statement or if card 3 has a square on the other side it disproves the statement. It does not matter if there is a circle or star on the other side of card 4, similarly it does not matter whether there is a square or triangle on the other side of card 1.*

Three of the participants correctly reasoned that only cards 2 and 3 needed to be turned and articulated the process linearly in that if you tried a certain card and the result held, you would go to the second card, and if not you had answered the question. An example of this reasoning is

> *If you turn card 3 and it has a triangle on it then you continue. If not it is false and you stop. If you turn over card 2 and it has a star it is false, if not the statement is true. It says every card with a star has a triangle, not every card with a triangle has a star.*

Stylianides & Stylianides further indicate adults usually correctly select Card 3, but also incorrectly select Card 4, focusing on the Q part of the statement. The data in Table 2 shows the number of times each card was chosen. Almost everyone correctly selected Card 3 and slightly more than one-half selected Card 2. However, as noted in Table 1 these two selections were not always the only two made as only 27.1% chose the two correct solutions with correct reasoning. Card 4 was also incorrectly chosen 41.7% of the time, while Card 1 was incorrectly chosen 20.8% of the time.

Table 2. *Number (and %) of times a card was chosen, n = 48*

Card 1	Card 2	Card 3	Card 4
10 (20.8%)	28 (58.3%)	43 (89.6%)	20 (41.7%)

Also of interest is the number of choices made for each of the 16 possible choices. This is shown in Table 3. Some information Table 3 has already been described but other items of interest include that 6 chose 3 only, focusing only on the P part of the statement. Of 28 of 48 correctly chose Card 2 and 27 of these also selected 3, but 11 of these also included another choice. They often gave reasoning; 6 participants felt all 4 cards must be turned over, focusing on P, Q, ~P and ~Q, although not all were needed.

Table 3. *Number and percent of times a choice was made, n = 48.*

Choice	Number of Times Selected	Percent of Times Selected
NONE	1	2.1%
1	1	2.1%
2	1	2.1%
3	6	12.5%
4	2	4.2%
12	0	0.0%
13	1	2.1%
14	0	0.0%
23	16	33.3%
24	0	0.0%
34	9	18.8%
123	2	4.2%
124	0	0.0%
134	0	0.0%
234	3	6.3%
1234	6	12.5%

Many of the explanations provided were partially correct, especially for those who chose Card 3, where the focus could be that P was true, hence needed to be checked. While only partially correct, the following explanation was in similar to those correct ones described earlier.

> *Since one side has a circle or a star, card #1 is not important to this statement. But we will have to check the other 3 to make sure a) #3 has a triangle on back b) #4 has a star on back c) #2 does not have a star on back. because we only need to verify star & triangle sides, not circles & squares. We are not verifying that if it isn't a star then there isn't a triangle.*

Discussion

It could be expected that those students enrolled in upper division mathematics or methods courses would be able to successfully answer this question and provide appropriate justification, but in these instances such was not the case. Most participants recognized that in a P --> Q situation, if condition P is met, then one must check if condition Q is met (Card 3). Less than half of the participants recognized Card 2 must be turned over because P --> Q is logically equivalent to ~Q --> ~P, and ~Q is met (Card 2). This calls into question why participants who are assumed to have the knowledge to accurately answer such questions were not able to satisfactorily use SMP 2 or 3 in this instance to disregard the context of the question, look at the structure of the question, and to provide a valid justification from known reasoning strategies.

This research has particular implications for precollege preparation in mathematics for secondary teachers. Wu (1999) makes the recommendation that in the mathematical preparation of high school teachers a change should be made from organizing courses that relentlessly look forward to graduate courses to organizing courses that spend more time looking back at the content of high school mathematics and have preservice secondary mathematics teachers develop a deep understanding of that content. This could be accomplished by examining topics in school mathematics that mimic situations like the question we posed and asking students to reason through them. By examining such questions in mathematics and mathematics education coursework, teachers and teacher educators have the opportunity to engage in rich discussions about mathematical logic. These discussions not only have the potential to illustrate ways in which the mathematics studied in college is specifically used in pre-college mathematics, but also may help consolidate the understanding of the mathematics being studied in college.

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AN EXAMINATION OF FACTORS IMPACTING COLLEGE ALGEBRA READINESS: PATHWAYS THROUGH DEVELOPMENTAL MATHEMATICS

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Many students entering higher education are under-prepared for college-level mathematics courses and require developmental mathematics. However, students struggle to complete the prescribed sequence of developmental courses. Reform initiatives suggest promising models for developmental coursework, yet insufficient evidence exists to support that these models are effective at scale. In this paper, we present a five-year longitudinal examination of a cohort of community college mathematics students and provide information as to what paths through developmental math were most successful. An accelerated format and initial placement in the developmental sequence were significant predictors of success in credit-level mathematics.

In the open access world of community colleges, students without college readiness in mathematics are afforded a second chance through developmental coursework. These courses do not satisfy any mathematics credential but serve to reinforce and strengthen algebraic skills. Often, a series of two to three courses will make up the required sequence for students that are underprepared. However, more and more students entering developmental mathematics are failing to progress to college readiness and a credit-bearing mathematics course. In a three year examination, Bailey, Jeong, and Cho (2010) reported that eighty percent of students entering a developmental math sequence do not successfully navigate through to a college-level math course, prompting leaders in the field to refer to current developmental mathematics sequences as the "graveyard of dreams and aspirations" (Bryk, 2012, p. 1).

As a result, community colleges are scrambling for solutions to this dilemma. With large numbers of underprepared students seeking community college as a gateway to higher education, the problem has drawn national attention. Innovation and creativity play a role as colleges seek inventive ways by which underprepared students can become college-ready in mathematics with increased passing rates and in less time. Options being considered as promising alternatives include self-paced, accelerated, and online/hybrid models. In this paper, an analysis of several potential pathways through developmental mathematics will be presented, with the intent to discover which models are most likely to lead to success in a credit-bearing mathematics course.

Related Literature

Although College Algebra was once considered the only entry into credit-level mathematics, much of the research now is focused on recognizing and justifying alternatives to College Algebra as viable mathematics courses for non-STEM students. Leading organizations in the field such as the National Council of Teachers of Mathematics (NCTM), the Mathematical Association of America (MAA) and the American Mathematical Association of Two-Year Colleges (AMATYC) have encouraged educators to stress quantitative literacy and statistics as opposed to algebra to better address the job-related skills needed for twenty-first century learners (Strother, Van Campen, & Grunow, 2013; AMATYC, 2006).

At the same time, very few promising practices have emerged in terms of course structures and course-taking patterns that will prepare students for a credit-bearing mathematics course – we call these course structures and patterns "pathways." Hern & Snell (2010) reinvented the long developmental sequence for math and piloted an accelerated one semester developmental course that allowed even the weakest math students to enroll in a Statistics course in their second semester. Students in this redesigned "Statpath" course outperformed students that had come through the traditional multi-class sequence, implying that for developmental course sequences, longer is not always better in improving student's skills. A massive course redesign at Cleveland State Community College (Squires, Faulkner, & Hite, 2009) proposed that a mastery-based self-paced model greatly increases the developmental math success rate and likelihood of students progressing from developmental mathematics to credit math courses.

Not all reform models are positive, however. In a recent study examining several promising alternatives to traditional developmental coursework, Cafarella (2013) cautions that acceleration is a best practice for some but not all developmental math students. Snell's innovative redesign efforts show promise, but are aimed to improve students' paths through Statistics, not College Algebra (Hern & Snell, 2010). In an examination of several modalities for developmental math courses, Keller (2014) found that technology-based modes such as hybrid (i.e., partially online) or fully online courses actually hindered success for developmental math students, echoing the findings of Zavarella & Ignash (2009) who reported that students in an online setting were more likely to withdraw from the course. The research base is accumulating evidence, but there is no clear preponderance on which educators can base instructional decisions. Additionally, no model has been studied at scale, and so educators are implementing many pathways through developmental mathematics in an attempt to find the solution to the crisis at hand. As well-known community college researcher Uri Treisman has stated, "…there is no silver bullet to fix developmental education. We need silver buckshot." (Treisman, 2013). Recent reform innovations promise to address the deficiencies for many underprepared college students, and in time, whether or not these promising practices hold the key to success in reforming developmental education will be determined.

In the present study, we investigate the following research questions: How are course structures in developmental mathematics courses, including starting course in the developmental sequence as well as accelerated, hybrid/online, or self-paced courses, associated with community college students' likelihood of reaching a credit-bearing math course? How are these structures associated with their likelihood of actually passing a credit-bearing math course?

Methodology

To investigate the paths that led to successful completion of the developmental mathematics sequence and credit math courses, a cohort of first-time-in-college (FTIC) students at a mid-sized community college system in a Southern metropolitan area was selected. The final sample included 595 students that initially enrolled in the Fall 2009 semester in any level of developmental mathematics. The demographic information related to this cohort is described in Table 1. Students were predominantly Caucasian and approximately one-third were of low socio-economic status, as measured by their eligibility for Title IV funding. Nearly half of the students in this cohort began the developmental sequence in Pre-Algebra, the lowest developmental course, while the rest began in Beginning or Intermediate Algebra. Students' first developmental course was usually determined by a placement test, such as THEA, Compass, or Accuplacer. Students were subsequently tracked for the 2009-2014 academic years, providing a five year glimpse at their enrollment patterns and course grades in mathematics. Variables related to the type of mathematics course they took, such as developmental/credit, self-paced, accelerated, or hybrid/online, were created and coded. Developmental courses for this cohort included Pre-Algebra, Beginning Algebra, and Intermediate Algebra. Entry-level credit math courses that the developmental sequence led to included College Algebra, Statistics, Quantitative Literacy, Trigonometry and Precalculus.

Table 1. *Initial cohort, Fall 2009 FTIC developmental math students (n = 595)*

	n	%
Race/Ethnicity		
Caucasian	257	43%
Hispanic	78	13%
African-American	45	8%
Asian	9	2%
Hawaiian/Pacific Islander	3	0.5%
Not Specified	203	34%
Gender		
Male	232	39%
Female	363	61%
Title IV/Pell Eligible	207	35%
Years since High School Graduation	Mean = 2.61 (SD = 6.33)	
First Developmental Class Taken		
Pre-Algebra	258	43%
Beginning Algebra	154	26%
Intermediate Algebra	183	31%

To analyze students' pathways, we coded successful completion of an entry-level credit-bearing mathematics course as a 0/1 dependent variable and employed logistic regression techniques to predict completion. Withdrawals and grades of F were considered failures, while a grade of A, B, C, or D was considered passing. We also coded whether students ever *enrolled* in a credit-bearing math class (regardless of whether they passed) as an additional 0/1 dependent variable. Several predictor variables were defined to describe the course taking patterns or pathways that students followed. First, accelerated courses are taught in a condensed or shortened time frame, including 8 week "fast track" courses and 5 week summer classes. Second, self-paced classes are those in which students progress through a series of assigned content modules by demonstrating mastery in order to proceed. Third, hybrid or online courses were combined to form a category of courses using technology for over 50% of the course delivery. If a student took at least one developmental math class that was hybrid/online, they would receive a code of 1 for this predictor; otherwise they would receive a 0. Coding for accelerated and self-paced predictors was similar. An additional predictor included which of the three developmental courses in the sequence students began with – Pre-Algebra, Beginning Algebra, or Intermediate Algebra. Although this variable could be considered an element of the students' pathway, practically speaking it is more of a measure of math background knowledge. The weakest students tended to place into the first class (Pre-Algebra) based on entry test scores, while the stronger students were placed into the third class (Intermediate Algebra). Additional student demographic data (Table 1) was added to the model as predictor variables to control for background characteristics and examine interactions. Due to the large number of missing values, race/ethnicity was excluded as a predictor in all models.

The logistic model is given by $\ln\left(\frac{p}{1-p}\right) = \alpha + \beta_i x$ where p is the probability of enrolling in or successfully completing a credit math course, α is the regression intercept, representing the logit for a student in the base reference category, and β_i represents the regression coefficients (slopes) related to each predictor variable x.

Findings

An initial examination of the cohort provided the following summary statistics (Table 2).

Table 2. *Overall course taking data for 2009-2014 (n = 595)*

Criteria	Number of Students (%)
% starting in Pre-Algebra that pass Pre-Algebra	186 of 258 (72%)
% starting in Beginning Algebra that pass Beginning Algebra	104 of 154 (68%)
% starting in Intermediate Algebra that pass Intermediate Algebra	124 of 183 (68%)
% pass first credit-bearing mathematics course	157 of 595 (26%)
% that enroll in a credit-bearing mathematics course	228 of 595 (38%)
% take at least 1 developmental accelerated	57 out of 595 (10%)
% take at least 1 developmental online/hybrid	110 out of 595 (18%)
% take at least 1 developmental self-paced	40 out of 595 (7%)

The first regression model examined whether enrollment in a credit-level entry course was predicted by course pathway or first developmental class taken. Odds ratios calculated in the logistic regression model were transformed to D-type effect sizes (standardized mean differences) using an approximation that allows for ease of comparison among logistic models (Chinn, 2000). Students enrolled in an accelerated developmental math course were more likely to eventually enroll in a credit mathematics course (Odds = 4.47, d = 0.83, p < .001), while students with self-paced or online/hybrid courses were approximately equally likely to enroll or not in credit mathematics courses (p = 0.48, p = 0.51, respectively). This makes sense given the rigorous nature of accelerated courses compared to the more traditional timeframe of a self-paced or online/hybrid course. Additionally, students enrolled in Beginning Algebra were more likely than Pre-Algebra students to enroll in credit mathematics (Odds = 2.14, d = 0.42, p = .001), and students in Intermediate Algebra were also more likely to enroll in a credit class than those in Pre-Algebra (Odds = 5.32, $d = 0.92$, $p < .001$) or Beginning Algebra (Odds = 2.58, $d = 0.52$, $p < .001$). This means that the closer a student starts in the developmental sequence to a credit mathematics course, the more likely the student is to enroll in the credit math course. SES was not significantly related to enrollment in a credit-bearing class ($p = .70$), and gender was marginally significant ($p = .07$) in that males were less likely to enroll in a credit-bearing class than females (Odds = 0.71, $d = -0.19$).

Interaction effects were not present when examining class format (accelerated, self-paced, online/hybrid) with first class enrolled or SES. However, an interaction did occur for online/hybrid and gender. When allowing for interactions, males were overall significantly less likely than females to enroll in a credit-bearing class than females (Odds = 0.60, $d = -0.28$, $p = .01$). However, males were more likely to enroll in a credit-bearing class if they took an online/hybrid developmental math class, compared to if they had not (Odds = 2.63, $d = 0.53$, $p = .04$). This suggests that while online/hybrid classes do not increase the likelihood of credit math enrollment overall, the odds do improve for males that enroll in online/hybrid developmental mathematics courses. This may relate to other research that has suggested that males have more positive attitudes towards computer-based learning than females (e.g., Vale & Leder, 2004).

The second model examined the impact of course format and student characteristics on passing a credit math course. As in the first model, students who took at least one accelerated developmental math course were more likely to pass an entry level credit math course (Odds = 4.20, $d = 0.79$, $p < .001$), while self-paced or online/hybrid courses did not increase the likelihood ($p = 0.47$, $p = 0.85$, respectively). The student's initial placement in the developmental sequence was again statistically significant. A student starting in Beginning Algebra was more likely to pass a credit mathematics course eventually (Odds = 2.06, $d = 0.40$, $p = .006$) than a student starting with Pre-Algebra, and a student starting in Intermediate Algebra was also more likely to pass an entry level math course (Odds = 4.38, $d = 0.82$, $p < .001$) than a student starting with Pre-Algebra or Beginning Algebra (Odds = 2.12, $d = 0.42$, $p = .003$). The interactions between course format and demographic variables were insignificant, with one exception. The interaction between accelerated courses and SES was negative, in that students with identified financial need that enrolled in accelerated developmental courses were less likely (Odds = 0.25, $d = -0.77$, $p = .0499$) to ever pass a credit-bearing mathematics course, compared to students who did not have financial need. This outcome suggests that while the accelerated course format was beneficial for students not in financial need, it was not quite beneficial ($p = 0.17$) for those who were in financial need.

Discussion & Conclusion

We analyzed the effect of course format and pathways on whether developmental math students enroll in and pass a credit math course. An examination of a cohort of FTIC students requiring developmental math coursework revealed that accelerated course options increase the likelihood of students completing developmental requirements and enrolling in and passing an entry-level mathematics course. Accelerated coursework may be a powerful but under-utilized option to move students successfully through the developmental sequence. However, interactions revealed that the benefit of accelerated developmental coursework for passing a credit-level course was not present for those who were financially in need. While a positive effect of accelerated coursework could be driven by stronger students tending to take accelerated coursework, our data actually suggests that weaker students who are starting earlier in the developmental sequence are more likely to enroll in accelerated courses. Both self-paced and online/hybrid formats did not increase the overall likelihood of enrolling in or passing a credit-level math course. However online/hybrid classes appeared to be beneficial for male students, who were less likely than female students to enroll in credit-bearing math courses.

Not surprisingly, the placement of a developmental student into one of three developmental courses also significantly impacted the likelihood of eventually enrolling in and passing a credit math course. Students starting in the first course, Pre-Algebra, were the least likely to ever complete a credit math course, with Beginning Algebra students more likely to do so, and Intermediate Algebra students even more likely to do so. While this may suggest that less developmental coursework is preferable, it is complicated by the differing math background knowledge of students who tend to be placed into each of the three courses.

In current work, we are adding in stronger controls for mathematics background knowledge using students' mathematics placement test scores – this will allow us to see if findings for accelerated courses and initial course placement are being driven by unmeasured knowledge differences. We also did not examine whether optimal pathways are different for students who move into a credit-level statistics course versus a credit-level algebra course (an important distinction as the developmental sequence is algebra-focused) – this is also a focus of current work. We are further adding more detail to the conceptualized pathways and considering additional factors such as the number of semesters required to complete developmental courses and the grades students achieve in these courses. We are also currently examining the impact of having to repeat mathematics courses due to failure or withdrawal, and the effect of course format and pathway given additional student variables such as language, years since high school, and full-time/part-time status.

Clearly, additional analysis is warranted, but these results have important implications for advisors of community college students and community college leadership and policy-makers. In the flurry of reform efforts to improve outcomes for developmental mathematics students, evidence is needed to determine the best course of action. We will continue to examine this issue and hope to contribute to the knowledge base that in turn impacts changes in course taking patterns and formats to positively impact students.

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SUPPORTING AN ELEMENTARY TEACHER IMPLEMENTING THE COMMON CORE STATE STANDARDS FOR MATHEMATICS

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This presentation describes and discusses an action research project where the researcher served as a participant/observer working with a fifth grade teacher in a suburban elementary school to improve her mathematics teaching. The project goal was to help the teacher negotiate the implementation of the Common Core State Standards for Mathematics (CCSSM). Efforts focused on analyzing the CCSSM, problematizing mathematical tasks, and nurturing the teacher's ability to foster discourse and alter the sociomathematical norms of the classroom.

This study was precipitated by a conversation between the researcher and her former student teaching mentor and colleague. The teacher, a veteran fifth-grade teacher-pseudonym Mrs. Larson, seemed overwhelmed and discouraged as her district was in the process of transitioning to the CCSSM (NGACBP/CCSSO, 2010) and adopting a new form of assessment. Although the state formally adopted the standards in 2010, in 2013, shortly *after* the school year started, the teacher's district decided to implement the CCSSM. This decision was imposed with such swiftness, that none of the teachers were prepared or had any professional development with the standards. Because of these changes, the teacher stated she felt as though she was drowning. She felt that she did not have the time or support to really understand and implement all the changes that were happening.

Furthermore, she viewed the stated goals and intended purpose of the CCSSM in a negative light perceiving them as being "dumbed down". The researcher offered to assist her with understanding and implementing the standards. From this start and subsequent discussions, the initial goals emerged. The teacher stated her goal as, "I want my students to leave my classroom with a deeper understanding of math, not just have the ability to memorize for a test." With this in mind, the researcher's initial goals were: (1) to improve the teacher's understanding of the CCSSM, and (2) change the way the teacher views math, teaches math, and, ultimately, the way her students learn math.

Theoretical Framework

Although many of the CCSSM standards are consistent with the previous state standards (Hanover Research, 2010), with any standards there is an underlying expectation that teachers may have to change their behaviors and instruction in the classroom. With the CCSSM, the impetus for transformation is more imperative. Gojak (2013), notes that, "teachers and administrators must have access to high-quality professional development, including opportunities to deeply understand the Standards for Mathematical Content and the implications for instruction of the Standards for Mathematical Practice."(p. 1). She further asserts that in order for these standards to be enacted to their full potential, teachers need an opportunity to truly understand them, learn how to use them intentionally, and have rich math problems and curricula that increase discourse within the classroom. The Conference Board for the Mathematical Sciences (2012) concurs in with this call. In a large-scale study, Garet, Porter, Desimone, Birman, and Yoon (2001) note that the characteristics of professional development that impact increases in knowledge and skills and changes in classroom practice: (a) focus on content knowledge; (b) provide opportunities for active learning; and (c) are coherent with other learning activities. These characteristics informed this study.

Method

This was an action research study where both the teacher and researcher/participant-observer/coach were co-participants involved in the cycle of planning, acting, observing and reflecting on classroom instruction in a deliberate and systematic manner; resulting in a public report of that experience (Perry & Zuber-Skerritt, 1992; Fischer, 2009). The classroom teacher and researcher collaborated over time to plan, analyze, and self-reflect on the instruction to enhance her teaching of mathematics. This on-going cycle of planning, acting, observing and reflecting was repeated throughout the study to guide decisions about interventions employed with the teacher.

Participants

Mrs. Larson was a female, 36-year veteran in the same suburban school district in the Pacific Northwest. She had taught various grades from first to sixth but has been teaching fifth grade for the last eleven years. Her class consisted of 23 students; 11 females and 12 males. The teacher worked closely with her two 5th grade colleagues. For math, the team planned ahead; jointly deciding the schedule of lessons from the *Math Connects* (MacMillan/McGraw-Hill, 2009) curriculum. They also assigned one teacher per week to choose the math homework for that week, aligning the homework to the lessons they were teaching in that time frame. The math homework consisted of worksheets included with the district's curriculum.

The researcher was a female, elementary-certified teacher with an undergraduate major in K-8 mathematics and one year of teaching experience including employing problem-centered learning (Hiebert, et al., 1997; NCTM, 2003) and discourse practices. She functioned as a participant-observer.

Data

Data collection involved field notes of teaching observations and discussions between the teacher and researcher. Formal discussions were also audio-recorded. Students' ability to communicate in math was formatively assessed through daily instruction and activity as opposed to using formal instruments. During observations, the researcher looked specifically for the mathematical language the teacher used, the questions she was asking the class, changes in body language of students or teacher, student responses, and the amount of time she allowed the students to struggle and persevere.

Initial Plan

On average, the researcher attended the math block in the classroom four days a week over the course of four months. The teacher and researcher met numerous times throughout this process. They met 1-2 times a week to discuss ideas and concerns, as well as, create lesson plans and math problems to use in the future. They also met or had brief conversations after lessons. This debrief discussed the student discourse, what was successful, whether the CCSSM was addressed and/or met in the lesson, and what could be improved before the next lesson. They also discussed the student learning that took place during the lesson. These meetings were important to the study because "analysis of lesson effectiveness both during and after lessons (NCTM, 2007) are essential factors in the growth and improvement of instruction" (Larson, 2012, p. 112). Throughout the entire process, teacher and researcher reflected and revisited strategies that worked for both her and her students and revised study goals as necessary to meet Mrs. Larson's changing needs.

Results

Initial observations of Mrs. Larson's teaching took place during the math block. On occasion, the researcher was also able to observe her teaching other content areas, such as science and social studies. The contrast was instructive. In her math block, the daily routine was to grade the homework from the night before first, then lecture new concepts or procedures, followed by guided practice before assigning homework; characteristic of traditional math classroom. The homework typically consisted of fluency practice from the prior day's lesson drawn from *Math Connects*, the district's curriculum. Each student would grade his or her own homework as the teacher quickly went through it on the document camera. Students who thought they had the correct solution would raise their hands, but most students did not. Typically, the same five to seven students raised their hands each time. After grading homework, students wrote the score and turned the papers into the math tub for gradebook entry. Mrs. Larson used math homework grades to pull students for further instruction and intervention. In these small groups, she usually pulled out related manipulatives and had students work through the procedures. She did not question them to find out what they knew; her goal was helping them find the correct answer or helping them learn rote procedures.

Body Language

Mrs. Larson felt that her strength and passion was in teaching science and social studies. Her love of these subjects, especially science, was very clear in her body language while she was teaching them. She got excited. She walked around the class. There was more inflection in her voice and she used her hands. When students asked questions or made conjectures, she took them in stride. She asked the students questions to make them think about their own thinking. She got the students to communicate what they knew. She exhibited passion. This passion was contagious to her students and others listening to her teach, myself included.

In contrast, none of these behaviors were evident in her math lessons. Mrs. Larson sat hunched over and closed-in behind the document camera. Her voice was lower, and she looked stressed. She went step-by-step through the teacher's guide; rarely straying from the text. When students questioned the algorithms and procedures that she was comfortable with, she was unable to address them. If she could not answer a question, a typical response was to repeat the rule or the procedure. She appeared to be curriculum bound. Despite being a veteran teacher, she clearly was uncomfortable with teaching mathematics. Research has shown that teachers act as a model to their students, so a teacher's view of mathematics, no matter how masked, can affect the attitude of the students (Charalambous, Pansoura, & Philippou, 2009). A teacher's tone, verbal and nonverbal cues, and body language all have a role in students' attitudes, success, and view of mathematics.

Perception of Mathematics

As they began working together, Mrs. Larson was asked how she felt about mathematics, how she was taught to teach math, and how she felt about teaching mathematics. She was extremely honest about her strengths and weaknesses in the classroom. She recalled that, as a student, she was taught in a "traditional math instruction" setting and there was little to no conversation about any math ideas. She was not asked questions to deepen her level of understanding, to make mathematical connections, to learn problem-solving techniques, or encouraged to examine math concepts. She also remembered that there were students that could just "do math" and other students that just "didn't get it." She often times felt as though she just could not or did not get it either. She felt that her personal strengths were in English, history, and science. Her college experience reinforced her perceptions of math. She took the minimum required math classes to graduate with her teaching degree, and her collegiate experiences with math were akin to her K-12 experience. It seemed that she may never even have thought in those early years, or in the years since, that there was more to math that just steps and procedures. Mrs. Larson subsequently confirmed this notion when she said, "I never even stopped to think that there was a deeper level to math than what I was taught."

The way that Mrs. Larson was taught math as a student naturally carried over into her mathematics teaching; traditional and procedural. She openly admitted that her understanding of mathematics was weak. Throughout our discussions, she said she felt unprepared to question her students, let them struggle productively, and learn to persevere. She took comfort in teaching the same math lessons year after year. She stated, "I have taught these lessons 500 times, so I know how to teach them. I know the questions that the students are going to ask." She felt like she understood the mathematics procedures she taught. She thought the correct way to teach her students was the same way she learned; just algorithms and rules.

New Goals

As we proceeded new goals emerged, to help Mrs. Larson understand how to create mathematical discourse in her classroom and to persevere in her own problem solving. A unit on operations with fractions involving like and unlike denominators became a significant learning experience for her. In her experience with fractions, as a student and a teacher, involved only memorized the mathematical rules of fractions, such as, "invert and multiply," "multiply straight across," "find the common denominator," etc. When asked to show and explain what $\frac{1}{2} \div 2$

meant, she was unable to do anything but invert and multiply. When asked why that worked, her response was, “That is the way I was taught. That is the way I have always done it.” In order for her to be comfortable teaching fraction tasks to her students, she would need to be comfortable with the fraction tasks herself before she could work on discourse. It became clear that it was necessary to take the additional work of building her conceptual understanding. A related realization was that, in order for Mrs. Larson to teach her students to persevere in mathematics and learn to explain and defend their reasoning, she too would need to persevere in her own learning. Pintrich and Zimmerman (as cited in Kramarski & Revach, 2009, p. 380) confirm that, in order for a teacher to create a classroom of self-regulated learners (SRL), the teacher too must have his or her own self-regulation in learning. They define SRLs as learners that are “good metacognition strategy users. They plan, set goals, select strategies, organize, self-monitor, and self-evaluate at various points during acquisition”.

The researcher had Mrs. Larson use mathematically rich tasks found in Van de Walle (2007). Some problems were worked on together at our meetings and some were given to her as “homework.” They both read specific chapters in this book and reflected on them when they met. The researcher encouraged her to solve the problems at least one way and, if possible, in as many ways as she could. This was very challenging for Mrs. Larson. There were times when she would just try to use the procedure she already knew to solve the problem. For example, when given the problem $\frac{1}{2} \times \frac{1}{2}$, she tried to just multiply straight across. Asked why she could multiply straight across; why that was a reasonable rule. She was unable to respond. The researcher restated the problem in this context, “You have to paint half of your garage. You have already painted half of the half you have to paint. How much of the whole garage have you painted?” She drew a picture of the garage. Using her picture, she was easily able to show me and tell me why the answer was $\frac{1}{4}$. She was then re-asked the original question of why she could multiply straight across. After a few blank stares and an “I don’t know,” she recognized the connection in the multiplication array in her picture. A huge smile lit her face. She was asked to defend this revelation several more times with different numbers, she decided that this would work for all multiplication of fractions problems. She said, “I have been using this rule for over 40 years, but no one has even shown me why it works.” The ensuing discussion acknowledged that she had not been shown how to do it. It was the use of questions and problem context that enabled her to make sense of it and to explain her reasoning. This is what she needed to do for her students. We discussed how this could impact student learning.

These types of interactions and insights occurred throughout the study as Mrs. Larson worked building her understanding; reaching a number of “ah-ha” moments. Sent home to solve some rich tasks that were candidates to use to with her students, she came back the next day saying,

Mrs Larson: *Ya know, when I was doing the problems, I was like Grrr! But then after I solved them, I felt really good!*

Researcher: *Have you felt good or proud in the past when you solved problems by using the mysterious rules*?”

Mrs. Larson: She looked at me and grinned. *Nope!*

She looked surprised by her answer, and this was a huge epiphany for Mrs. Larson. She said then that she wanted her students to get the same satisfaction and understanding she gets from persevering in solving these mathematically rich tasks.

This was an important experience for Mrs. Larson, and transferred to the way she wanted her own students to learn for two reasons. First, Mrs. Larson learned the advantage of discovering the solution on her own. She found power in her ability to have made meaning of the mysterious rules, and recognized how that will play a role in her ability to recall it in the future. Secondly, the perseverance she exhibited as she solved problems became a motivator for her. She felt proud of herself after conquering these challenging tasks. Unprompted, she noted that she did not have these same reactions while doing rote procedures. Asked to reflect on what she learned for her teaching and her students learning, she repeated several times that she wants her students to feel this same success and pride.

This, too, was also a good lesson and reminder for the participant/observer that, for the work together to be meaningful, the teacher needed to be empowered to struggle with the problem of selecting tasks, building discourse, making decisions. Choosing problems and modeling problem-centered lessons for her was not practical for building her ability to enact the CCSSM on a long-term basis. Mrs. Larson needed to experience the struggle and power of teaching through problems for it to come to life for her as the teacher. In other words, in much the same way as she worked on the mathematics tasks, she had to learn problem-centered teaching and discourse by doing.

Discussion

The main goal of this research was to assist a veteran fifth grade teacher with implementing the Common Core State Standards for Mathematics in the classroom, along with, helping her create rich mathematical discourse. However, the plan altered to spend less time on the CCSSM *per se* as had been originally planned. Instead, focus necessarily shifted to spending significant effort building Mrs. Larson's conceptual understanding of mathematics in the context of locating, developing, and solving rich problems to build a basis for creating mathematical discourse with her students and, by extension, furthering her understanding of the CCSSM. Mrs. Larson's concerns, struggles, and needs are by no means atypical. In this country, children are taught math from k-6+ by teachers who are not considered, or even expected to be, math experts (Cai, Ding, & Wang, 2013).

This limited study provides an example of how implementation of the CCSSM may be problematic, unless teachers have a firm understanding of mathematical ideas. Moreover, they are likely to struggle developing that understanding without assistance building their mathematical knowledge. With modest support, Mrs. Larson did deepen her content knowledge and gain insight into the standards. It is possible that the CCSSM may come to be viewed as another set of failed standards if teachers do not have effective professional development, the opportunity to collaborate rather than simply plan, and access to resources for math support. Teachers, in general, strive for excellence and want their students to leave their classrooms with a higher level of understanding, problem solving skills, and a passion for learning. However, if these teachers aren't aware that there is more to math than "black and white memorized procedures," and "mysterious rules" like Mrs. Larson, how can they be expected to educate our students differently and appropriately?

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SIGNIFICANTLY TRAUMATIZED CHILDREN IN THE MATHEMATICS CLASSROOM

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This exploratory study investigated the connection between students who had experienced significant childhood trauma and subsequent mathematics learning. The purpose of this action research project was to observe the behavior and responses of traumatized middle school students immersed in a problem-centered mathematics learning environment that was intentionally designed to address their needs. Preliminary results suggest positive growth was observed in each student's willingness to explore and learn mathematics, ability to determine what they were capable of, ability to communicate their needs, and apparent motivation.

This paper describes an action research study conducted by a middle school mathematics teacher working at an alternative school in a large urban school district in the Pacific Northwest. This school serves students from diverse backgrounds who have been gathered based, not on their age or academic level, but on their behavior. These students have been through countless unsuccessful interventions at previous schools and, ultimately, been assigned to this alternative school. The principle characteristic that these students are likely to share is having experienced some form of significant childhood trauma.

In this project a traumatized learner (TL) is defined to be a student who has experienced abuse or neglect, witnessed domestic violence, experienced environmental violence, or had a loss of a caregiver through divorce or death (Cook et al,2007). TLs tend to act with impulsivity, aggression, defiance, withdrawal, and unrealistic perfectionism (Cole et al., 2005). Childhood trauma and is a form of disability that can manifest itself through flagrant misbehavior and an apparent general refusal to participate in the learning environment. Trauma hinders a student's ability to build relationships or organize their thought patterns. This is consistent with Quinn et al (2000) who noted that students who are emotionally distressed are at a natural disadvantage compared to their counterparts. To address this, the alternative school has a large therapeutic component to it that aims to deal with the trauma and has a strict structure to help control and modify misbehavior.

The genesis of this project was the teacher's desire to better serve his students who exhibited a general distaste for math and an apparent unwillingness to work. Drawing on his background as a graduate student in mathematics education, he performed a limited trial that attempted to engage these students through problem-solving. Initially, when a problem was presented to the students, they exhibited their usual resistance. After several days of leading students through the process of problem-solving, they started to catch on. Several weeks into the trial, the difference was very noticeable as the students began to look forward to the daily challenge. They were able to persevere through difficult problems for the entire class period and began using each other for potential ideas. They used their failed attempts to inform their next strategy and explained their thought processes to the teacher and other students. There was evidence that they continued to discuss problems from the class outside of the classroom. Most importantly, their task avoidance and flagrant misbehaviors abated.

These effects made the teacher wonder what it was about these math tasks that drew such desirable outcomes from these students and what would happen if applied more generally. Exploring the literature, no studies could be found that specifically addressed TLs and mathematics learning. Also, the literature on problem-centered instruction does not address TLs. Moreover, the studies related to TLs primarily focused on strategies for addressing the behavioral issues TLs bring with them from their childhood traumas that affect the way they learn rather than specific content. In short, there is a significant lack of research in this area.

In order to understand how to educate TLs, it is helpful to understand their impairments. There is a consensus on the following broad domains of impairments observed in TLs (Cook et al., 2007, Perry, Pollard, Blakley, Baker, & Vigilant, 1995). TLs are affected by variety of outside influences that can impede their ability to participate appropriately in the classroom. They have difficulty processing situations, and their reactions can appear incoherent and unorganized. Over time, traumatic stressors have impeded the natural growth of behavior patterns, cognition, identity, and regulation of emotion in these children. As a result, these students may feel ineffective, helpless, deficient and unlovable, and may have no idea what they are realistically capable of accomplishing. The difficulties each child experiences are often reflected in his or her habits and behaviors in the classroom.

A typical TL may have been socialized in a way that impedes his or her ability to communicate. TLs tend to have little experience processing situations verbally and are more likely to enter the classroom in a state of anxiety, causing them to have difficulty communicating their needs or accepting new information. This can lead to a decreased ability to problem-solve affective situations (Cook et al., 2007). Essential skills such as goal setting, anticipating consequences, or anticipating rewards are difficulties for traumatized students. Emotionally disturbed students often have disorganized and inconsistent upbringings. The environments in which they have grown up may have been unsafe and/or restrictive, resulting in a lack of experience in independent exploration. When children do not explore, they have difficulty realizing that actions have consequences or that they can influence what happens to them, leaving them unwilling to try and displaying dependent behavior. The unstable environments to which they are exposed leave them depleted of motivation and other internal resources for engagement, resulting in difficulty with affect regulation along with reduced ability to organize and explore (Perry, Pollard, Blakley, Baker, & Vigilant, 1995). These challenges can cause the student to act instead of plan and may lead to inappropriate classroom behavior.

TLs, like all students, need to be met where their needs are. Tailoring classwork to the abilities and interests of the students more often leads to success. A common obstacle of TLs is their likelihood of having severe gaps in knowledge. Careful attention should be paid to address the needs of these students, because they can also lack the ability to organize their thoughts for learning. Teachers can build strategies for "learning how to learn" by incorporating strategies that teach them how to take notes, track their progress, manage their time, and organize their materials rather than having those skills be a pre-requisite for assigned work (Cole et al, 2005). Perhaps the most important thing to remember is that these students have difficulty with motivation and affect regulation (Quinn et al., 2000). Keeping students motivated is likely to lead to engagement. Every TL has different challenges, but teachers can increase the likelihood of motivation by empowering the students with choice, using technology, and using hands-on experimentation.

Cole et al (2005) describe academic strategies that have been shown to be effective for teaching TLs. Fostering a positive learning environment and attention to the relationship between educator and student is important. The teacher should consider the social and emotional state of the child during interactions (Quinn et al., 2000). Important strategies for fostering achievement in TLs include: (a) Attention to task difficulty. Students with emotional disturbances from traumatic events have experienced failure so often that they are likely to have many more gaps in their knowledge and require tasks appropriate to their level, (b) Engagement. Traumatized students are more likely to misbehave in the classroom when they are not engaged or are overwhelmed. This can be mitigated by breaking longer presentations and assignments into shorter segments, giving students the time that they need to finish a task, following low-interest activities with high-interest activities, and helping students learn from their mistakes (Quinn et al., 2000), (c) Attention to motivation. Motivation can be improved by building on students' interests, empowering students with choice, exploring technology, and using hands-on experimental learning to increase motivation and provide a more consistent motivational context for all learners (Cole et al., 2005), and (d) Focusing work assignments. Many TLs lack study and organizational skills. These students need to learn how to learn in the classroom. They need to learn how to take notes, track their progress, manage time, and organize their materials so that their backgrounds for learning are more consistent with that of their non-traumatized peers (Cole et al., 2005). Teachers need to create assignments that will help students develop these often missing skills while also supporting their learning of the content they are studying. The strategies above, while designed to serve TLs, are effective for all children. However, traumatized children reside within classroom populations, and a classroom approach that considers their needs can be helpful to both TLs and to children who have not experienced trauma.

Problem-centered instruction as an instructional approach in that is distinct from a “traditional” or basic skills instruction in mathematics. A problem-solving approach requires that learners understand problematic situations sufficiently to draw upon or devise one or more strategies, based on the current schema the learners have, to solve the problems. Further, students are expected to support their answers through mathematically appropriate reasoning and justification. Problem-solving focuses on how learners reach solutions, often with multiple avenues to arrive at solutions (Lesh & Zawojewski, 2007; Lester & Kehle, 2003). Further, we take Coy’s (2001) stand that learners' schema, communication, willingness to experience and explore new things, and being open to look at things from more than one perspective are all a part of the problem-solving experience. We anticipated that problem-solving would increase students' mathematical curiosity and proficiency by challenging them to draw on the mathematical understanding that they have (their schema), at times reinforcing or supplementing their knowledge of mathematical fundamentals through applying them, and thereby increasing their motivation to participate in mathematics. The goal was for TLs to persevere in and attend to details in solving mathematical tasks without withdrawing or misbehaving due to their previous lack of exploration, communication deficits, and perceived lack of ability caused by their trauma.

The hope is that students grow in their ability to explore, organize and communicate their needs, perceive themselves as capable mathematically, and grow academically by repeatedly experiencing the rising to, and meeting of, a challenge.

The purpose of this exploratory study was to begin to address the lack of knowledge on the mathematics leaning needs of TLs by implementation of a problem-centered approach (Lesh & Zawojewski, 2007; Lester & Kehle, 2003) in the teacher's class. The following questions guided the teacher during the investigation:

- Does a problem-centered approach improve the ability of TLs to learn mathematics?
- What responses, affective and cognitive, are observed in traumatized learners when they are taught mathematics using a problem-solving approach?

Method

The participants of this study were four male middle-school students (two White, one Native American, and one Hispanic) drawn purposefully from a math support class consisting of nine males. The math support class supplemented their primary (i.e., regular) math class. This was a typical class size and composition at this alternative school. Students in this class had varying mathematics abilities, ranging from 1^{st} grade to 8^{th} grade level equivalencies. The four students (subjects D, B, S, BR) were selected because they were the most recent arrivals and it was highly unlikely they had experienced a problem-centered math class previously. The students in this study understood they were being observed for a research study but did not know the details or purpose of the project, in order to garner authentic results. In contrast with the requirements in the students' primary math class, no specific curriculum or math concepts were required to be explored in the support class. The instructor was allowed to choose what and how to teach the support class. He chose to use a problem-centered approach that focuses on problem-solving exclusively. The instructor/researcher was a white male teacher in his first permanent teaching position who was pursuing a master's degree in mathematics education.

This action research study used qualitative, action research methods (Hubbard & Power, 1993, Stringer, 1999) to address the questions and analyze the data. The researcher/participant-observer planned, acted, observed and reflected on classroom instruction in a deliberate and systematic manner. The data included field observations and notes regarding the TLs social behaviors and responses to the problem-centered course work, as well as, artifacts documenting their academic progress. Analysis of the data and reporting on each subject involved first reflecting on the information provided about and initially learned from each subject as he first joined the classroom. Analysis included rereading through all of the field observation notes of each subject; daily reflections on student work, behavior, and actions; and considered additional insights and data gathered on each subject as the weeks of the project progressed. This initial analysis was returned to, reflected upon, and refined on an on-going basis as the study progressed.

TLs were presented with a variety of problems that had no specific mathematical focus other than problem-solving itself. Problems chosen ranged from easy to difficult and addressed various mathematical ideas the students needed to experience and including algebraic, geometric, and various number sense concepts. Easier problems were strategically placed in-between more difficult problems in order to moderate any anxiety the TLs might feel from working on the problems.

Results

Subject D made large strides in his academic behavior during the quarter. Initially, he was confrontational, aggressive, and became easily frustrated. After the first week, he experienced some success and began to buy into the process. He became able to take feedback and eventually sought it out. Later, he showed signs of truly enjoying the problem-solving process. He asked for additional challenges, his self-talk became more positive, his signs of frustration started to fade, and he went from accusing staff of purposefully “pissing him off” to laughing about his mistakes. Evidence suggests this process made D a better problem-solver. Towards the end of the year, his primary math teacher related positive stories about D's improving quality of work.

Subject B exhibited an unwillingness to explore that was rooted in a fear of being viewed as unintelligent by his peers. His fear was irrational but understandable as half the students in the class were continuing from the previous quarter. They had already gone through a process of frustration with problem solving and were now clearly more comfortable with the process. When B saw these other students, students whom he thought he was smarter than, doing well on these problems, he became fearful of looking “stupid” and became disruptive and dismissive. From the beginning, he showed some ability to complete problems but would opt out if there was any perceived danger of not being able to complete the problem. At the end of February, the instructor had a long discussion with him and his therapist about the class. He finally admitted his behavior was covering his fear of failing in front of others. B was informed that every student in the class had had some difficulty adjusting, and a plan was devised for how he could deal with his challenges. After this intervention, he became more cooperative, more willing to explore and to fail in front of his peers while solving problems. B began to ask for help and face difficulty in public without completely shutting down. He continued to mask his self-perceived shortfalls and anxieties but in less disruptive ways. This continued throughout the study. By the conclusion of the project, B had become more resilient in his other classes and has since increased the score on his placement test. His primary math teacher has reported a more participation from B and his grades improved.

Subject S was initially very defiant. He refused to try, was unwilling to explore, could not communicate his needs when faced with difficulty, and viewed himself as not capable. Early into the observation, he showed signs of interest in the content when he saw a possibility of completing a problem. When he was successful, he showed a willingness to explore, attempted to communicate his needs, and believed in his capabilities. After the first week, he began to attempt tasks without any negative behaviors most of the time; indicating that he thought he was capable of solving the problem. As long as the teacher was near, he would attempt to communicate his difficulty. However, he had difficulty doing so appropriately. He would yell out in a disruptive manner for help and, if he did not receive it soon enough, he would misbehave. Over time, he was able to attempt strategies, work with others, and accept feedback with fewer misbehaviors. This indicated that his willingness to explore was increasing. He accepted and asked for challenges indicating that his interest in math and motivation was increasing. The teacher noticed that when S missed a day, he lost a little bit of progress. If he was suspended or in In-School Suspension (ISS) the previous day, it was likely he would be less willing to start the problem or persevere to a solution. It is inconclusive whether S may have increased in his mathematical ability. The experience he received during this project exposed him to several concepts and he showed modest growth in his ability to solve problems. It is telling that, on a number of occasions, S arrived late after he made the decision to self-process out of ISS so he could attend this particular class, indicating that he saw some sort of value in it. He rarely attended his primary math class.

Subject BR struggled throughout this project. He was the last student to join the class due to being in ISS because of his behavior. At the very first sign of difficulty, he would argue and question the validity of the course. His perfectionism may have caused him to misbehave to mask fears that the task was too difficult. During the first week of his attendance, some positive changes were observed. There was an increase in his willingness to attempt problems and reason through them. This indicated he was willing to try and explore, communicate his needs when faced with difficulty, and reveal that he was having difficulty in front of peers.

However, he exhibited a recurring cycle of setbacks. After a day in ISS or a day with the instructor absent, he often regressed, exhibiting fear of failure, unwillingness to explore, and difficulty accepting feedback and ideas. When he experienced some success, several days of growth would follow. He showed signs of belief in himself by attempting difficult problems and persevering through difficulties by expressing his needs. Towards the end of the project, he was expelled from school. When he returned he was placed into a different class. It is possible that BR would have experienced more growth if he could have attended the class with more consistency and exposure to problem solving would have made him a better math student. He has subsequently advocated for more difficult math classes and his placement scores have increased.

Discussion

The evidence from this limited study suggests that the problem-centered class does have a positive effect on TLs' learning of mathematics. Affectively, limited but positive growth was observed in each student's willingness to explore, ability to self-assess mathematical capabilities, ability to communicate learning needs, motivation improvements, and positive affective responses such as enthusiasm and positive attitude. When observed in a more traditional mathematics classroom, the behavior of TLs contrasted significantly with the problem-centered study classroom, where they were more engaged, positive, and communicative. The limitations of this study are numerous: limited duration, small sample size, inconsistent attendance on the part of participants, lack of standard measures of growth in subjects' mathematical or academic ability. However, there is modest evidence that the increased motivation and engagement observed would continue to yield growth. This suggests that further study on TLs and mathematics learning is warranted.

This study investigated a group of TLs' experiences with problem-centered learning over the course of an academic term but does not begin to answer all that needs to be learned about TLs and mathematics learning. The benefits of problem solving have been well-documented with a number of populations. However, the dearth of research about TLs and mathematics is concerning. Further research is needed to shed light on how to help TLs learn mathematics and not slip through the cracks. In addition to a more rigorous follow-up study, a number of further research questions are suggested: Can TLs compensate for and strengthen their affective deficits through problem-centered learning? What teaching strategies help TLs learn mathematics? How do TLs benefit from problem-centered learning? How might TLs grow in a more traditional mathematics classroom? These questions and many more await exploration.

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DEVELOPING PRODUCTIVE DISPOSITION IN STRUGGLING MATHEMATICS STUDENTS

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This study examines the development of productive disposition in a class of high school students with histories of being unsuccessful in mathematics courses. Students in this class showed substantial improvement in the elements of productive disposition during the course of an academic year.

Students who are behind in math tend to keep falling further behind. Students who are unsuccessful in math come to believe that they cannot be successful. Students who do not master mathematics early in their academic career often develop a conception that mathematics simply does not make any sense. For many students, school mathematics becomes a downward spiral full of self-fulfilling prophecies of failure and unproductive beliefs about the mysterious nature of mathematics and mathematics learning. Results from this study indicate that it does not have to be that way. It is possible, even when students have been unsuccessful for years, to change the narrative and help students develop a more productive disposition towards mathematics.

Background

Adding It Up (National Research Council, 2001), in describing 5 intertwined strands of mathematical proficiency, defined *productive disposition* as a "habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy" (p. 5). This disposition plays an important role in the development of other strands of mathematical proficiency. For example, students' conceptions of mathematics as sensible are linked to their ability to and likeliness to strategize about mathematical problem solving (Wong, Marton, Wong, Lam, 2002), that is, in the development of the strategic competence strand of mathematical proficiency. Similarly, the adaptive reasoning strand, in which students are expected to provide explanation and justification, relies on a view of mathematics as sufficiently sensible that such explanations and justifications exist.

Unsurprisingly, both self-efficacy and student beliefs about the usefulness of mathematics are correlated with students' achievement in mathematics. Students who are not successful in mathematics are somewhat more likely than their peers to believe that math is not useful (Kadijević, 2008) and substantially more likely to doubt their own abilities in mathematics

(Kadijević, 2008; Multon, Brown, & Lent, 1991). Students' conceptions about the nature of mathematics also appear to be correlated with student achievement (Schommer-Aikins, Duell, & Hutter, 2005; Stage & Kloosterman, 1995; Steiner, 2007). For example, Schoomer-Aiken, Duell, & Hutter (2005) found that "Both general epistemological beliefs and mathematical beliefs appear to influence mathematical performance and overall academic performance" (p. 301). The research in this area seems to bear out the experience of many teachers, that students who are less successful in mathematics also tend to have less productive dispositions towards mathematics.

This less than uplifting picture naturally leads to the question about whether it is possible to break this downward cycle and what it would take to help students change unproductive dispositions and improve mathematics achievement. This study examines a high school class that serves as an existence proof that change is possible and provides some insights into how to help all students get back into a position where they can be successful in mathematics.

The initial idea for the study began, not with looking for productive dispositions, but with a teacher wondering why some students in her high school classes seemed to engage in mathematics very differently than others. Some students were more willing than others to jump into unfamiliar mathematical territory, make connections between mathematics and life situations, and keep working at a problem until they succeeded. Although this attitude might not be unusual among high-achieving mathematics students, this teacher also saw it in a number of her students with long histories of being less than successful in mathematics. This study began with the identification of a phenomenon, a need to know more about that phenomenon, and a hope that this knowledge could lead to ideas for how to recreate the phenomenon in more math classrooms.

The Study

The data for this study comes from one class of beginning algebra students in a small, rural school in northern New England. More than 10 years before data collection for this study took place, this school began offering special "supported" sections of some of its mathematics courses. Placement in these classes is by permission only and is limited to students who have experienced difficulties with mathematics in the past or who have a documented learning disability related to mathematics. The classes were designed to be co-taught by a mathematics teacher, a special educator, and an instructional assistant. Classes ran for 90 minutes every other day for an entire academic year. The first year of data collection in this study occurred when the students were enrolled in an Algebra 1A course in which they would learn the material from the first half of a traditional Algebra 1 course. All 18 students enrolled in the class agreed to participate in the video portion of the study. Thirteen agreed to be interviewed. Most of the students were freshmen and most were enrolled in a supported Pre-Algebra class during the previous school year. Many of them experience difficulties in multiple subject areas and are more likely than the school population as a whole to face disciplinary action for breaking school rules.

The teacher involved in this study, Mr. Wingate (a pseudonym), has a bachelor's degree in mathematics and approximately 35 years of experience teaching high school. He has been teaching supported sections of mathematics classes for more than 10 years. He also teaches the school's AP Calculus classes and a class in desktop publishing.

Data for the first year of this study consists of classroom observations of 10 classes, video recordings of these 10 classes and of most class meetings throughout the academic year, student interviews, teacher interviews, interviews with the special educator, and classroom artifacts such as student work, quizzes, and worksheets. Most of the information presented in this paper is from two weeks early in the fall and two weeks in late spring of the same academic year.

Data Analysis

Because of the recognized difficulties using self-reporting to measure such constructs as beliefs, conceptions, and dispositions (Munby, 1982), I used evidence from classroom observations as the primary source of data for examining students' dispositions towards mathematics. This data was supplemented by focus-group style interviews in which students were encouraged to discuss their level of agreement with 10 beliefs statements. These statements, adapted from survey instruments used in other studies of conceptions of mathematics, included statements like: "A math problem can always be solved in different ways" (Brown, et al., 1988) and "It is not important to understand why a procedure works as long as you get the right answer" (Kloosterman & Stage, 1992). Classroom data were analyzed using the Students' Conceptions of Mathematics as Sensible (SCOMAS) framework (Grady, 2013). This framework (see figure 1) consists of action-oriented indicators developed from the literature on students' conceptions of mathematics.

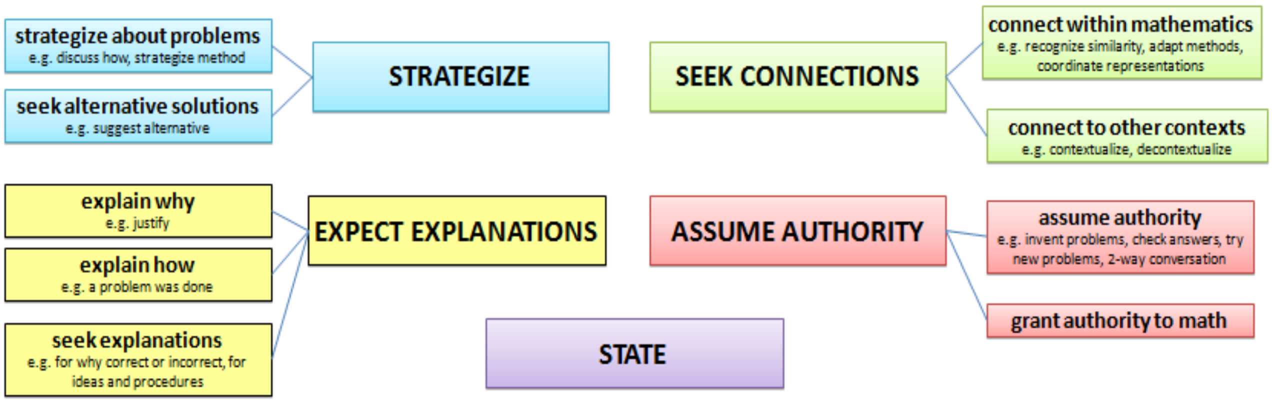

Figure 1. SCOMAS Framework (Grady, 2013, p. 144)

These indicators are organized into a framework to allow coding of students' conceptions along 5 strands: strategizing, seeing connections, explaining, mathematical authority, and stating. This coding presents a rich picture of the dimensions of students' view of mathematics as sensible. In addition, because the seeing connections strand includes students making connections between mathematics and real-world situations, the framework provides a way to code data about students' view of mathematics as useful. Both the mathematical authority strand and the strategizing strand contain indicators such as "Students being willing to try to solve a problem for which they have not been taught a procedure" and "assume mathematical authority" which provide insights about students' sense of self-efficacy in relation to mathematics. In addition to coding data using the SCOMAS framework, classroom and interview transcripts were searched for student statements and incidents related to self-efficacy and the usefulness of mathematics. Statements from student interviews were used as corroborating evidence for trends seen in the classroom data.

Results

Analysis of classroom interactions using the framework of indicators provided substantial insight into the ways in which students in this class conceive of mathematics as sensible and useful and about the students' belief in their ability to engage in mathematics. Students in this class demonstrated conceptions of mathematics as sensible across the four major strands of the framework. Due to space considerations, I will discuss just two of the strands: expecting explanations and seeking connections.

Students in this class demonstrated that they believed that mathematics is sensible enough that it can be explained and justified. The most substantial growth in this strand is in the area of justifying mathematical statements. Although there were as many incidents of justifying in the fall as in the spring, almost all of the justifications during the fall observation were prompted by the teacher. During the spring observation, students initiated all except 4 of the 21 episodes of justifying. By the end of the course, students had also come to seek explanations for classroom mathematics. Students began in the fall simply accepting what they were told about mathematics with no coded incidents of seeking explanations. During the spring observations, students sought explanations for procedures nine times and justifications for mathematical ideas four times. In several of these incidents students demonstrated remarkable persistence in continuing to ask questions until they were satisfied with the explanations that they received.

So strong were students' conceptions about the importance of explanation in mathematics that several groups had difficulty in the interviews imagining doing mathematics without knowing why a procedure worked. When asked about their level of agreement with the statement, "It is not important to know why a procedure works as long as you get the right answer" (Kloosterman & Stage, 1992) students uniformly disagreed. One student stated, "I feel like there's no point in doing it if you don't understand why you are doing it" while another stated, "You have to know how it works in order to do it again." In addition to these practical reasons students provided for why it is important to know why things work one student provided another reason. He stated, "I wouldn't enjoy math if I didn't know why, like what I need to do to get the answer, like why I was doing it." Students in this class had come to view mathematics as sensible enough that, not only were there explanations for things, these explanations and justifications were in integral part of what it meant to do and enjoy mathematics.

Students in this class also grew in their expectation that mathematics should be connected both to other mathematics and to other contexts. During the two weeks of the fall observation, there were only 5 incidents in which students made connections between mathematical contexts. All of these incidents involved noticing surface-level similarities. For example, the teacher put two problems on the board: a) 5/8 + 3/2 and b) 5/8 – 3/2. The class had already gone over the solution to part a, converting the problem to 5/8 + 12/8. At this point a student noticed that the numbers were the same and, thus, they did not need to redo the initial steps of the problem. Although the student clearly connected two mathematical problems, the connection did not require the student to go beyond what was immediately in front of him. By contrast, during the spring semester there were 13 coded indicators related to students seeing connections within mathematics and all but one of these went beyond simply noticing surface-level similarities between mathematical objects. For example, when examining a scatter plot of data one student observed that it seemed to have a positive slope. The student made this connections even though the concept of slope had not been discussed in this context; indeed, it had only been discussed in the context of graphs of lines and, at this point, no best-fit line had yet been drawn.

In addition to the change in seeking connections within mathematics, there was a dramatic change in the ways that students made connections between mathematics and real-world contexts. During the fall semester there were 2 incidents coded as making connections to other contexts; during the spring semester there were 19, almost all unprompted. Although some of these connections were as basic as comparing shapes in a mathematics problem to familiar objects, more than half required students to go beyond just visual connections of how to model or directly interpreting representations. In these incidents students often extended the mathematical ideas and sought to reason about how they might apply. For example, when the class created a bell-shaped curve to represent the distribution of heights across a human population and discussed how that is related to the fact that this particularly tall teacher could not buy pants at a local store, one student make the connection to business decisions stating, "I just figured out why they order more of one thing than another – because they use the average. The business orders more of what customers are ordering." Although there were very few indicators during the fall observations that students looked for or saw connections between mathematics and real-world contexts, by spring it was an element in all class discussions. In fact, if real-world connections were not introduced, or were not persuasive enough for the students, students asked about them.

Discussion

Students came into this class with a documented history of low achievement in mathematics classes. Given the recognized correlation between elements of productive disposition and mathematical achievement it is reasonable to expect that these students began the course not particularly convinced that mathematics was sensible or useful or that their hard work could result in success in mathematics. Their tendency, by the end of this course, to expect explanations and justifications and to make connections between mathematical contexts demonstrates that they have developed a conception that mathematics is sensible, coherent, and connected. Further, their tendency to make connections between mathematics and real-world contexts and their insistence on meaningful answers to the question, “where will this be useful in real life” argue that they see mathematics as sensible. Students’ sense of self-efficacy in relation to mathematics, although less evident in the SCOMAS framework, is hinted at in the prevalence of unprompted explanations, justifications, and connections made by the students during the spring observation. These students appear to have developed a belief that what they have to say about mathematics is valuable and useful. They have learned that they are capable of participating in the mathematical discussion.

Although the literature points to a correlation between the elements of productive disposition and achievement in mathematics, it is likely impossible to determine the direction of causality. It is reasonable to suppose that a poor disposition leads to poorer achievement and that lower achievement leads to a poorer disposition towards the subject. Thus, students with a history of low achievement in mathematics are likely to get caught in a downward spiral. This study presents evidence that, even as late as the beginning of high school, it is possible to change students’ dispositions towards mathematics and, hopefully, break the downward spiral.

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MOMENTUM: BUILDING CAPACITY FOR CHANGE THROUGH CONNECTIONS

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According to the U.S. Department of Education National Assessment of Educational Progress for Tennessee (U.S. Department of Education, 2014), there was no significant increase in fourth-grade students' mathematics scores for almost a decade. The goal of Momentum: Building Capacity for Change through Connections was to increase student achievement by increasing elementary teachers' capacity to teach mathematics in a STEM-centered environment using children's literature. This professional development program, funded through Tennessee's Race to the Top grant, took a problem-solving approach to learning mathematical content and pedagogy. This paper addresses the efforts to deepen content knowledge of participants.

Rationale and Theoretical Framework

In March of 2010, Tennessee received a federal Race to the Top award of 500 million dollars to be used to implement a comprehensive four-year reform plan in education. This plan, First to the Top, addressed a wide range of issues which included changes in mathematics education. According to the 2014 U.S. Department of Education National Assessment of Educational Progress (NAEP) report, these changes were greatly needed. From 2003 to 2011 there was little significant increase in Tennessee fourth-grade students' mathematics scores. In 2011, the average score was 233 compared with the national average of 240.

Momentum: Building Capacity for Change through Connections was an 18-month professional development project funded through a First to the Top grant to Austin Peay State University (APSU). It was a joint project of the Department of Mathematics and Statistics and the Department of Curriculum and Learning, beginning in August 2011. The goal of *Momentum* was to increase student achievement by increasing elementary teachers' capacity to teach mathematics in an environment centered on science, technology, engineering, and mathematics (STEM) connected to children's literature. Project activities were aimed at the following objectives.

1. Deepen elementary teachers' content knowledge of the Common Core State Standards for Mathematics (CCSSM) through problem solving.
2. Broaden elementary teachers' pedagogical content knowledge by making connections to children's literature and science and by incorporating appropriate technology.

3. Strengthen teachers' understanding of the role of STEM in developing numeracy.
4. Deepen students' understanding of the core concepts of algebraic thinking, measurement, and data analysis.

This paper addresses results with respect to Objective 1, specifically the impact of the project activities in deepening participant understanding of what it means to know and do mathematics.

Throughout the project, problem solving served as a vehicle for increasing participants' content knowledge as well as strengthening their confidence in their ability to do mathematics. Adoption of the Common Core State Standards (NGA & CCSSO, 2010) was a major focus of the First to the Top plan. In the CCSSM Standards for Mathematical Practice, there is strong emphasis on problem solving.

The theoretical framework for the problem solving approach was grounded in the work of Duckworth (1991), whose theories on teaching and learning were based in part on her work with Piaget. In an interview in 1991, she spoke of teachers' need to "see their own minds get engaged, see their own confusions, their own tentativeness, their own excitement" (Meeks, p. 31).

In his overview of the history of research on mathematical problem solving, Lester (1994) supports this metacognitive nature of actually doing mathematical problem solving. More recently, in *Mathematical Education of Teachers II*, the Conference Board of the Mathematical Sciences (2012) recommends a strong emphasis on problem solving.

> All courses and professional development experiences for mathematics teachers should develop the habits of mind of a mathematical thinker and problem-solver, such as reasoning and explaining, modeling, seeing structure, and generalizing. Courses should also use the flexible, interactive styles of teaching that will enable teachers to develop these habits of mind in their students. (p. 19)

Most *Momentum* participants reported that they learned mathematics through skill development and algorithms. This project was aimed at providing a supportive problem-solving environment in which they might develop the habits of mind of a mathematical thinker.

Method

Thirty participants were recruited in teams from seven schools in four Middle Tennessee school districts. Of these, 27 completed the entire project. Most participants were either mathematics or science K-6 classroom teachers. However, one was a general resource teacher and another was an English Language Learner teacher who worked extensively with mathematics students. At the time of recruitment, only one K-6 school in the university service area had a faculty that was more than ten percent non-white. Although recruitment was vigorous, only four project participants were non-white.

A problem-solving environment is not just an environment where problems are posed. *Momentum* was designed to provide an environment that supported communication, collaboration, exploration, and sustainable collegial relationships along with needed resources for teaching and learning. A series of daylong workshops served as the centerpiece for the professional development. Participants attended 17 eight-hour professional development days spread over 18 months. Five of these workshops were part of a weeklong summer academy. The other workshops were held on Saturdays during the academic year. In addition, shorter online workshops dealt with specific topics such as reviewing curricula, using Geometer's Sketchpad™ and spreadsheets, and viewing videos of elementary classrooms where students were involved in mathematical problem solving. Often the teams from particular schools completed the online workshops together and submitted their work as a group.

During the daylong workshops, participants solved mathematics problems emerging from children's literature and from real life situations. In ongoing journals, they recorded their problem solutions. The problem solving journals were organized using a Mathematical Thinking Record adapted from a model developed at the Education Development Center (Driscoll, Zawoajewski, Humez, Goldsmith, & Hammerman, 2001). In this model, teachers record the problem, their own solutions, the solutions of others, the content addressed in the problem, and reflections on the experience, attaching student work when the problem is used in the classroom.

Participants planned lessons based on the problems they solved, connecting the lessons, where appropriate, to topics from science. They then taught those lessons and shared student work in subsequent workshops where they collaborated in grade level groups to create a bank of lessons and materials. For example, after reading *The King's Chessboard* (Birch, 1993), the participants solved the problem of finding how many grains of rice would be on a chessboard after starting with one grain of rice on the first square and doubling the number of grains for each subsequent square. They then adapted lessons for their particular grade levels, taught the lessons in their classrooms, and brought student work samples to share. This work in the classroom served to deepen participants' understanding of the value of problem solving, but review of student work samples also stimulated further reflection on their own problem solving strategies.

Many of the problems posed in the workshops involved connections to art, social studies, and science. Since a national presidential election was in progress, there were many opportunities to discuss the mathematics involved in polling, in voting, and in being a responsible citizen. Books such as *Grace for President* (DiPucchio, 2008) provided literature connections. Some of the most successful workshop activities integrated science in a variety of ways. For example, after reading *Who Sank the Boat?* (Allen, 2007), participants engaged in a study of measuring volume by displacement, an exploration involving balancing hanging levers, an investigation using a simple balance as a measurement tool, and creation of a balanced mobile.

To support participants' work in their classrooms and their participation in a larger community of like-minded educators, each received membership in the National Council of Teachers of Mathematics as well as membership in the Mid-Cumberland Reading Council. Both of these organizations support critical thinking as a major component of student learning. Materials provided included a set of children's literature books along with a book of suggested problem solving lessons tied to the books in the set. Teachers could choose the set appropriate for their own grade levels. Each participant also received a copy of Geometer's Sketchpad™ software. Participants had access to the curriculum materials in the APSU STEM Center and the APSU Department of Mathematics and Statistics, where the mathematics education courses are housed. APSU provided tuition for one graduate course for each participant, either MATH 5120: Contemporary Programs in K-12 Mathematics or RDG 3040: Expanding Literacy across the Content Areas. Additional resources were provided through these courses.

As a culminating activity, the project provided support for travel expenses, membership, and registration for each participant to attend the Tennessee Mathematics Teachers Association annual meeting. Several of the teachers presented at the conference, sharing teaching strategies, connections to literature, sample problem solving lessons, and student work samples. For most of the participants, this was the first time they attended a conference on teaching and learning mathematics.

In addition to the activities and content assessments, the First to the Top administrators required pre- and post-interviews with teachers, pre- and post-online assessments of attitudes and practices, and video taping of three lessons by each participants. These were assessed by an outside evaluator, and project developers were not provided with details of the results.

Evaluation and Findings

Evaluation of participants' content knowledge was a primary requirement of the First to the Top funders. Each grant project was required to create and administer a 25-question assessment instrument as a pre-test and post-test. To elicit as much detail as possible about participants' approaches to problem solving, project developers created an open response assessment related to three content strands: algebraic thinking, measurement, and data analysis. Many of the questions had multiple entry points and could be solved at various levels of sophistication. For example, a problem might be solved using trial and error, or it might be solved by developing a general rule or by applying previously known information. A four-point rubric was developed to assess the correctness of the solution to each question as well as the level of mathematical sophistication of the response, yielding a possible perfect score of 100.

Based on the varied mathematical backgrounds of the participants, a wide variation in responses was expected. Some teachers had only begun their teaching careers and had more recent experiences with mathematics courses and mathematics assessments. Others had been teaching for over ten years and had not been in a testing situation for many years, especially with mathematics other than that involved in the grade level they taught.

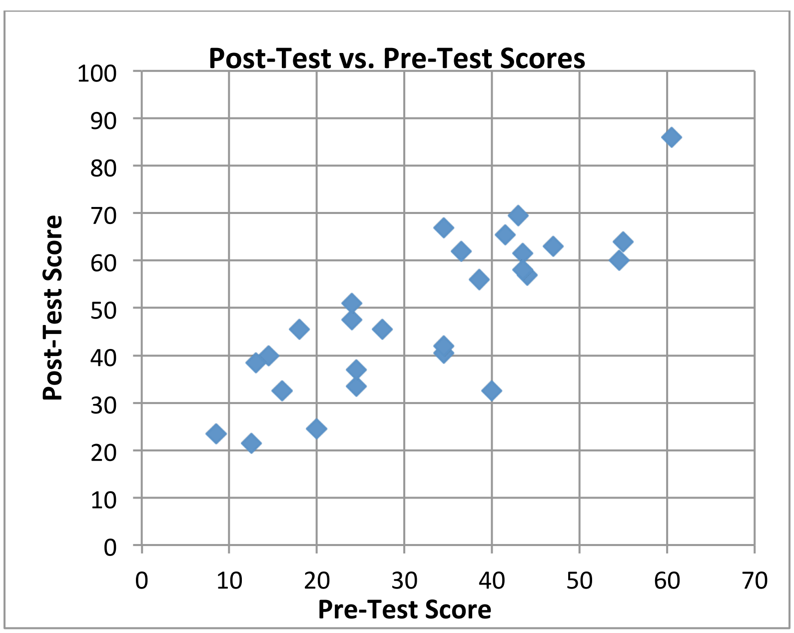

Figure 1. Post-test scores versus pre-test scores for individual participants.

Figure 1 shows the relationship between pre- and post-test scores for the 27 participants who completed the project. On the pre-test, the mean score was 32.5 with a standard deviation of 14.3, indicating the expected variation in responses. On the post-test, the mean score was 48.3 with a standard deviation of 16.1. A matched-pair t-test indicated the difference of 15.8 points, a 48.6 percent increase, to be significant ($\alpha < 0.01$).

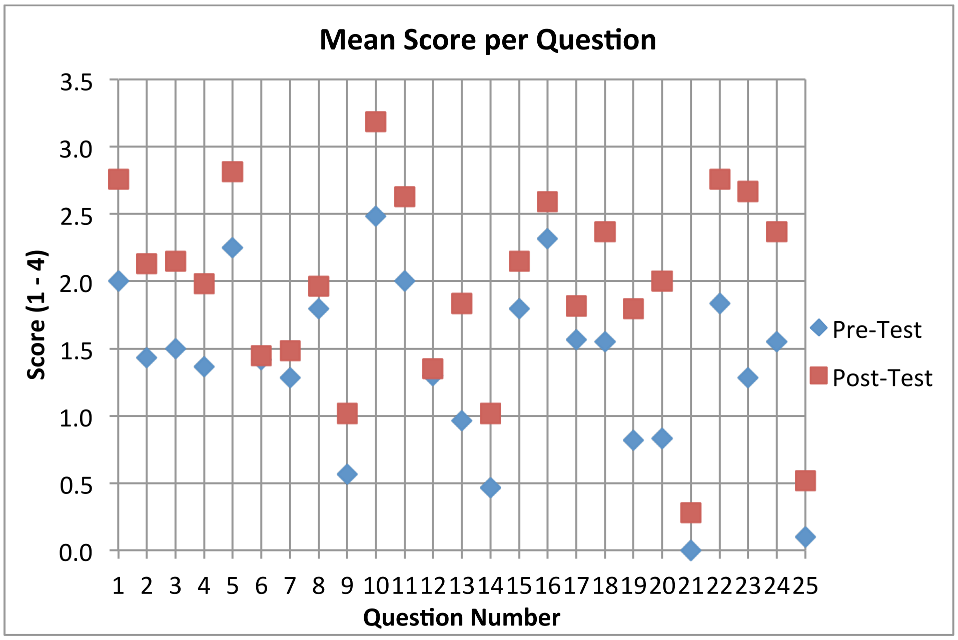

Figure 2. Mean responses to individual questions.

Figure 2 shows the mean pre- and post-test scores for each question on the assessment. For most questions, the mean scores improved. Questions 6 and 12 are notable exceptions because there was no change for either of these. The average question score increased from 1.4 to 2.0, an increase of 0.6 that was statistically significant ($\alpha < 0.01$). In qualitative terms, this would be an average increase from a novice level to a developing level of sophistication.

Question 6: In the equation $y = 4x$, if the value of x is increased by 2, what is the effect on the value of y?

Question 12: Write a mathematics word problem for which $3 \div \frac{1}{2}$ would be the method of solution.

As might be expected, most errors for Problem 12 involved dividing by 2 instead of by $\frac{1}{2}$. However, project staff expected to see more correct solutions to Problem 6, since it could be solved fairly easily by trial and error.

The questions for which the average responses were most improved were Questions 19, 20, and 23. In problem solving sessions, measurement and proportional reasoning were heavily emphasized. Improvement for these questions may be related to that work.

> Question 19: A rectangular pool 24 feet long, 8 feet wide, and 4 feet deep is filled with water. Water is leaking from the pool at the rate of 0.30 cubic foot per minute. At this rate, how many <u>hours</u> will it takes for the water level to drop 1 foot?
>
> Question 20: Suppose the pool in the previous problem is twice as long, twice as wide, and twice as deep. Will it take twice as long for the water level to drop 1 foot? Explain your answer.
>
> Question 23: A machine for producing Choco-nuts gives 5 choco-nut hearts for every 4 chocolate bars and $\frac{1}{2}$ cup of nuts put into the machine (nothing is lost in the machine). An order comes in for 48 hearts. How many bars and how many cups of nuts would be needed to fill the order?

Except for Question 12, which was based on number and computation, when data were disaggregated by content strand, participant scores in each category showed statistically significant improvement ($\alpha < 0.01$). Raw score gains were 0.7 for algebraic thinking (12 questions), 0.7 for measurement (four questions) and 0.6 for basic statistics (eight questions).

When participants first started the *Momentum* project, most did not have confidence in their problem solving skills. If a problem had several steps to be accomplished or several pieces of information, they seemed less likely to attempt to solve it. The improvement for these questions might be attributed to individuals simply taking time to analyze the information given and persisting in seeking a solution. This may be a result of their improved confidence with problem solving, something that project staff noticed in review of the problem solving journals and observation of the activities in the workshops. However, the overall improvement in scores seems to indicate a deepened understanding of at least some components of the mathematics content.

Throughout the project, most participants willingly engaged in all activities and workshops, often rearranging personal Saturday activities to accommodate for attendance. Even those who struggled with the mathematics or science worked through the problems with the help of their teams and eventually succeeded. When it was time to review student work samples, there was always a wide variety of samples at varying grade levels. Teachers took on leadership roles as they shared their activities with the rest of the group.

The *Momentum* project provided opportunities for teachers to form a community of learners bound together by common mathematical experiences. Although the state evaluator did not carry out a formal assessment of student achievement, in presenting their student work samples, participants reported high levels of student interest and achievement on the tasks and problems. On a state level, First to the Top was a success. In 2013, Tennessee fourth grade mathematics scores on NAEP averaged 240 (U.S. Department of Education, 2014), the greatest gains of any other state or jurisdiction.

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NEVADA MATHEMATICS PROJECT: EVOLUTION OF A STATEWIDE PROFESSIONAL DEVELOPEMENT PARTNERSHIP

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A framework for developing a statewide professional development mathematics initiative that involved collaboration among multiple agencies is presented. The data collected ranged from documenting the design decisions made and the impact of the professional development. The framework revealed the need for a jointly negotiated vision, co-creation of knowledge, negotiating format and delivery of the professional development, willingness to learn, and adaptability of professional development to local context. Collegiality influenced the design decisions that were made. These design decisions impacted the nature of tasks and delivery of the professional development.

Theoretical Framework

Designing professional development to improve teacher content knowledge and student learning is a challenge because the Common Core State Standards for Mathematics (CCSSO, 2010) attends to cognitive demand which is not representative of previous state standards (Cobb, & Jackson, 2011). Cobb and Jackson (2011) state that the feasibility of implementing the CCSSM standards as intended by the developers would "depend on schools' and districts' capacity to support the requisite teacher learning and on whether states are able to support schools' and districts' development of this capacity" (Cobb & Jackson, p. 2010, p.185). Furthermore, they point out that research should focus on the process of implementing the CCSSM. This paper documents the evolution of a professional development team that wrote a Math and Science Partnership grant to support teachers statewide to implement the CCSSM standards by deepening their content and pedagogical content knowledge. Math and Science Partnership grants are intended to improve teacher knowledge through collaboration between mathematics educators, mathematicians and school districts. Therefore, this type of collaboration can provide opportunities for developing capacity to support teachers. However, further research is needed to explore what constitutes effective collaboration and how individuals from diverse communities such as institutions of higher education, school districts, and professional development regions can work together to design effective professional development to support teachers' implementation of the CCSSM standards. Each partner comes from different institutions and therefore interacts within their own community of practice. People who are part of a community of practice participate in activities through mutual engagement, negotiate meaning through a joint enterprise and possess a shared repertoire of tools (Wenger, 1998). This means that people within a community of practice have developed ways of interacting and thinking with each other that might be unique to that particular community of practice and unfamiliar to outsiders. Wenger (1998) points out that when people in a community engage in practice they form close relationships and ways of participating that outsiders cannot easily enter. When a group of individuals from varying backgrounds form a professional development team, they do not automatically become a community of practice as defined by Wenger (1998). The authors of this paper jointly wrote a Math and Science Partnership grant. In this paper, we examine the following research questions: How did the project team evolve to design and deliver the professional development to support teacher understanding of the CCSSM? How did the evolution features contribute to the design of the professional development? What impact did the professional development have on teacher content knowledge?

Method

The Nevada Mathematics Project is a statewide collaborative that includes private, charter and public schools across the state. Over 126 (grades 3-8) rural and urban teachers across the state participated in the project. The goal of the project was to provide teachers with professional development on the Nevada Academic Content Standards based on the Common Core. Algebraic thinking and fractions domains of the CCSSM (2010) were the focus of the professional development. The professional development team consisted of mathematics educators from two different institutions of higher education within the state, two mathematicians, regional trainers, and district leaders. The project team travelled to four sites across Nevada that were located in different geographic regions across the state. Practically every county in Nevada was represented in the grant. A week-long professional development institute took place during the summer at each of the four sites with three follow up sessions during the academic year. A design research (Lamberg & Middleton, 2009) approach was used in delivering the professional development. In other words, the PD team made conjectures on tasks that could support teachers' understanding of content. The tasks were modified based on how teachers responded. Yin's (2014) single-case study research methodology was used. Case study research methodology is useful for understanding questions relating to "how" and why things occur. Multiple data sets were collected as part of the larger study that included video recordings, surveys and documentation of artifacts created during the professional development and planning sessions. The study and the analysis are still on going. For the purpose of this report, we present analysis of the pre and post content test data and analysis of field notes and artifacts created during project meetings. Pre and post math content tests were administered at the beginning and end of each summer institute. Field notes were recorded during planning meetings, teacher professional development sessions and the debriefing that took afterwards. The project team members took turns to record the field notes. For this analysis, the field notes data were coded and analyzed for themes related to the evolution of the professional development project team and its impact on the design and delivery of the professional development using the Strauss and Corbin (1998) Constant Comparative method. The pre and posttest was a 9 item test of mathematics content. Each test item was scored based on a 3-point scale. The average score for each pre and posttest item was recorded.

Results

Teacher Content Knowledge Pre and Post Test Results

The Pre and Post Teacher Content test data revealed growth in teacher content knowledge related to the CCSSM domains of algebraic thinking and fractions.

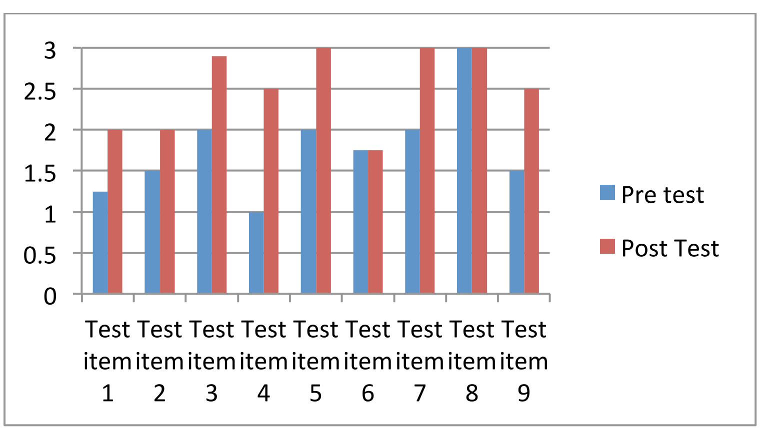

Figure 1. Pre and Post Teacher Content Test

Developing a Shared Vision

The writing of the proposal involved developing a shared vision. Jointly developing the proposal represented an integration of different perspectives and the formation of a joint vision. Initially, different project team members had different perspectives of how the professional development should be designed. Developing a shared vision involved listening to each other's perspective and mutually agreeing and negotiating a shared vision and goals. The challenges that the team experienced included having time to meet and geographic distance. This was overcome by having several smaller meetings within different communities of practice with phone and e-mail conversations. Conversations and written feedback took place based on the proposal draft that was shared between project team members. Each person was able to respond, question, shape, design and redesign the proposal until it was mutually acceptable for all parties. This written document was refined and re-written multiple times. This process involved articulating and developing a shared vision.

The initial stages for developing the proposal involved creating a vision and goals for the project. The latter stages involved logistics. When the project team met for the 2 day planning meeting after the grant was funded, it became critical to re-examine the vision and goals. This process involved designing the structure and the format of the PD. Here, the higher education team shared expertise and work, and mutually participated in designing goals/vision for the project.

The team wrote the vision and goals on a whiteboard, talked about the meaning of these goals and how they should be implemented in the design of the professional development. It was during this process that a generative shared understanding as a project team started evolving. The project team discussed the project goals and wrote down a set of outcomes to drive the design of the professional development, as outlined in the following field notes captured by the lead Principal Investigator (PI) (5/14/2014):

- Teachers will value content knowledge and conceptual understanding through classroom application.
- Emphasis on content knowledge and developing classroom applications to increase student outcomes.
- Student outcome: Student mathematical thinking via math practices and classroom learning. Develop conceptual and procedural fluency. –

The initial process of writing the proposal was agreeing upon goals through electronic means and small group conversation. However, a shared understanding was not reached until the whole team met together to design the professional development. At this point, the team started to work together on a shared goal and this resulted in the formation of a professional development team that was becoming a community of practice.

Explicitly Defining Each Person's Role within the Project to Build a Coherent Team

The professional development team was made up of mathematicians, mathematics educators, district trainers, regional trainers and district leaders. Each person contributed a unique set of expertise and knowledge. Articulating each person's role within the project and also the strengths, resources and knowledge they brought to the table became important for negotiating and defining each person's role within the project. Each person's role was clearly defined in the grant proposal. For example, the district trainers provided the team with insight about the needs of the local district. Their role was to recruit teachers and provide input to shape the design of the professional development to meet the needs of the local context. A timeline with learning goals, tasks, timeframe and individual responsible was created. These determinations were outlined in field notes taken by the PI (5/14/2014) as follows:

> All team members contributed to the format and the design of PD. We discussed how to sequence topics, what kinds of task should be presented, whether content and pedagogy should be taught separately? It was decided that one person will take a lead presenting different sections; however the team will join in as it made sense during the presentation. Pedagogy will be integrated with content.

When the team started to work together, the team also knew about each other's expertise and how they intended to participate in the project. As a result, each person's expertise and ideas were valued and honored as the vision was being prepared to be placed into action. Otherwise, a large team could lead to disconnected, unfocused professional development without clear goals. Furthermore, unclear roles could potentially lead to misunderstanding and miscommunication within the project. Not only was it important to understand how each other's expertise would contribute to the project, it became important to think about each team member's respective work community, and how this work would be recognized. Therefore, it was important to address how the work in the project would be mutually beneficial and recognized in each team member's work context.

Addressing Challenges of Scale

One of the largest challenges in Nevada is geography. The distances are great between rural communities and the urban areas. Communication is a challenge. Furthermore, districts and professional development regions value local control of their professional development. In order to address these needs, the team decided that it would be best to honor the local control by creating four different PD sites for the professional development that was representative of local control while maintaining a coherent vision. Furthermore, teachers from surrounding counties travelled to these sites. Therefore, it gave teachers across the state access to attend the professional development without burdensome travel. The district curriculum leaders aligned this training with the district/regional visions of how this project fit with their goals for supporting teachers to implement the Nevada Academic Content Standards. A "trainer of trainers" model was visualized for this project so that these teachers in the future could support other teachers.

Unpacking the Common Core Standards to Design the Professional Development

Unpacking the standards and coming to a consensus of what they mean shaped the design of the professional development. The team spent two days figuring out what the standards meant through examining related research, resources and discussion. Discussions took place on how to structure the professional development. Questions arose if the teachers should be separated for grade band groups such 3-5 and 6-8. As outlined in following field notes taken by the PI (5/14/2014), the group decided that it would be mutually beneficial for the teachers to understand the progression of standards:

> Discussed format of PD - After a lengthy discussion, the team decided to work collaboratively to design the whole PD instead of dividing into sub groups. Rationale: Liked - listening to each other's collaborative perspective.

The higher education team listened to district leaders and regional trainers about the needs from the field. The district leaders and regional trainers shared what they thought worked well and what did not work well in their local contexts. The CCSSM standards became the anchor in which the PD team sequenced and planned the activities for the institutes. The design decisions included who would teach which part of the standards, as well as types of tasks that would help teachers understand the standards. Through this process, the team developed a shared understanding of the intent and meaning of these standards. The team started to evolve as a *learning community* when the conversations shifted from logistics to outcomes. The initial question that was explored was "What do we need to do in order to get the project going? The questions shifted to: "What content should the teachers understand? What pedagogical skills should be integrated, and how do we sequence the PD to optimize teacher learning?" Further discussions took place with regard to the meaning of design research and how it was going to influence the design decisions. Basically, a conjectured learning trajectory and tasks were laid out during the planning meeting. These tasks were modified and adapted during the sessions.

Relationships: Development of Friendships

Human relationships became an important part of the success of the project. The continuous four-week training was intense. It required travel, and a combination of physical and mental efforts. During this time, the team travelled together, got to know each other as human beings and was engaged in laughter. The relationships that formed became critical to develop collegiality: giving each other honest feedback, learning about each other's personality and style. Furthermore, it afforded communication and developing shared mathematical language and unique ways of interacting. A significant aspect to note was that by the end of each week, the team was exhausted but elated at the same time. The learning, growth and friendship became important to working together for the subsequent year.

Conclusion

The project made an impact on teacher content knowledge of the CCSSM standards. The professional development team made up of many partners began to evolve as a *learning community* when they jointly communicated about the meaning and intent of the CCSSM standards and how to support teachers. The PD team developed characteristics of communities of practice identified by Wenger (1998). Cobb and Jackson (2011) pointed out that a systematic approach is needed to implement those CCSSM standards at a classroom level. The collaboration provided opportunities for the partnership to think about and communicate about the nature of systematic support could be implemented in the schools. Further research is needed to ensure that this systematic support will be realized in school settings. Figuring out how to support teachers to implement the Common Core Standards (CCSSO, 2010) is a complex process. The themes that emerged may be helpful for other projects attempting to do similar work.

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SUPPORTING K-10 TEACHERS' INSTRUCTIONAL CHANGES TO PROMOTE THE STANDARDS FOR MATHEMATICAL PRACTICE

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This manuscript describes two cohorts of teachers' instructional changes through the lens of the Standards for Mathematical Practices (SMPs). These teachers participated in a yearlong professional development program targeting the SMPs. Videos of their pre- and post-professional development programs were examined using a SMPs-focused protocol. They offered more opportunities for students to engage in the SMPs after the professional development experience than before the professional development. We connect this impression with ways to effectively foster elementary, middle, and secondary teachers' SMP-focused instructional practices through professional development.

Related Literature

Standards for Mathematical Practice

Many teachers are reevaluating their instruction because of the Common Core State Standards for mathematics (CCSSM; Council of Chief State School Officers [CCSSO], 2010). The CCSSM are composed of content standards, Standards for Mathematics Content (SMCs) and practice standards, Standards for Mathematical Practice (SMPs). The SMPs offer characterizations of behaviors and habits that students should demonstrate while learning mathematics. The *Principles and Standards for School Mathematics* (NCTM, 2000) and *Adding it Up* (Kilpatrick, Swafford, & Findell, 2001) guided the descriptions of the SMPs.

It is clear from literature that teachers' instructional emphasis of the process standards is not occurring often (Hiebert et al., 2005). Such emphasis is connected to promoting students' mathematical proficiency, as described in the CCSSM. Initial research reports about CCSSM implementation suggests that K-12 teachers are struggling to make sense of the SMPs (Bostic & Matney, 2014b; Olson, Olson, & Capen, 2014) much less weave the SMPs into their everyday instruction on the SMCs (Bostic & Matney, 2014a). These findings call for professional development to enhance teachers' understanding of the SMPs and support them to design and actualize instruction that makes the SMPs a part of their mathematics teaching. The purpose of this paper is to build upon the current literature base as a means to discuss K-10 mathematics teachers' instruction, specifically focusing on the ways they provide students' opportunities to engage in the SMPs and its influence on instructional mathematical discourse.

Professional development

A metaanalyis of professional development (PD) suggests that there are some key features to designing effective inservice teacher education (Guskey & Yoon, 2009). Two of those five features include (a) PD activities that encourage teachers to adapt a variety of practices to a content area rather than encouraging a set of best practices and (b) PD activities that encourage teachers to try ideas in their classroom. Boston (2012) details how focusing on implementing worthwhile tasks during a yearlong PD enhanced secondary teachers' knowledge, which in turn influenced their instructional practices. For example, after the yearlong PD they were able to identify elements of tasks with high cognitive demand and concurrently selected more tasks with high cognitive demand for their own instruction. Improving teachers' ability to select worthwhile tasks is not the only way to impact their instructional outcomes (Boston & Smith, 2009); supporting them to establish an effective learning environment and sustain mathematical discourse between students are also necessary to maximize students' opportunities to learn (NCTM, 2007).

In this study, two yearlong projects were conducted in a Midwestern state to prepare teachers to implement the CCSSM. We aim to explore how teachers' instruction changed to support students' engagement in the SMP and attempt to connect their growth to the PD project. Our research question was: How does teachers' instructional encouragement of the SMPs change during the PD? Further, we wondered how teachers' changes might be related to three central areas of this PD: learning environment, worthwhile task, and discourse. We examined K-10 teachers pre- and post-PD mathematics teaching specifically looking for specific instructional actions that are connected to the SMPs.

Method

Context of the Professional Development

We focus on K-10 teachers' experiences as influenced by two yearlong grant-funded professional development programs. Cohorts of K-5 and grades 6-10 (i.e., Algebra 2) mathematics teachers volunteered to be a part of a one-year program during 2013. Teachers met four times for four-and-a-half hour spring sessions between March – April. They met during the summer (June - July) for eight 8-hour days and then again in the fall (August – November) for two face-to-face meetings lasting for four-and-a-half hours each. Teachers were provided with numerous online assignments that were intended to facilitate further online interactions between

March – October that might support teachers' understanding of the SMPs. Generally speaking, the aim of the PD projects included (1) making sense of the SMPs, (2) exploring inquiry through three broad areas consisting of worthwhile tasks, mathematical discourse, and appropriate learning environments, (3) implementing classroom-based tasks that aligned with the CCSSM, and (4) increasing mathematical knowledge and understanding. Teachers read and reflected on their own mathematics instruction, as well as the instruction of others who were implementing the standards. Teachers read and discussed chapters from NCTM books (e.g., Mathematics Teaching Today [NCTM, 2007]) and completed various assignments including reflective journaling, writing, enacting, and reflecting on CCSSM-aligned mathematics lessons, and solving rich mathematics problems.

Participants

This project served 36 K-10 teachers across one Midwest state. Twenty elementary and 16 secondary mathematics teachers participated. Teachers came from urban, suburban, and rural school districts in a Midwest state.

Data Collection and Analysis

Teachers were asked to design, enact, and videotape one lesson during the spring when the PD began and again in the fall, after the PD concluded. Depending on the grade level and the local school context of the teacher, the videos were as short as 25 minutes and as long as 65 minutes. Since our study focused on ways that teachers supported students' engagement in the SMPs during instruction, we investigated the videotapes as a means to best report any instructional changes made during the PD program. Such analysis approaches have been used in similar studies such as Boston (2012) and Boston and Smith (2009).

Data analysis required two parts. The first part involved watching the videotapes and reflecting on instruction using a protocol focused on the ways that teachers' instruction supported engagement in the SMPs. Two mathematics education faculty as well as five mathematics education graduate students watched the videotapes and conducted the analysis using a protocol developed by Fennell, Kobett, and Wray (2013). It provides look-fors that link mathematics instruction with behaviors and actions that are associated with the SMPs. For example, three aspects were used for the first SMP: Make sense of problems and persevere in solving them. They included (a) Involve students in rich problem-based tasks that encourage them to persevere in order to reach a solution, (b) Provide opportunities for students to solve problems that must have multiple solutions, and (c) Encourage students to represent their thinking while problem solving (Fennell et al., 2013). While there may be other aspects indicative of SMPs, the protocol provides an evidence-based framework for examining mathematics instruction using the SMP lens. Next, we compared our coding observations with one another. When there was a difference in codes, a third coder watched the video and discussed his/her findings with the initial coders. Discussions ended when coders agreed that there was sufficient evidence related to a look-for.

The second part of data analysis focused on making sense of the data to answer our research question. We intended to quantify changes in the number and type of instructional opportunities related to the SMPs. This was accomplished by examining our evidence in two ways. The type and frequency of instructional opportunities related to each SMP were categorized. We summed the total number of indicators for each SMP during pre-PD instruction and compared that grand total to the grand total of indicators for all SMPs seen in post-PD instruction. Summing across all indicators transformed the ordinal data into continuous data thus the sums were examined using a paired-samples t-test. Our continuous data set met the expectations for conducting a t-test (Agresti & Finlay, 2009). Then, we compared the number of indicators observed during pre-PD and post-PD instruction for each SMP using a chi-square test. Finally, we explored the changes in instructional opportunities related to the SMPs across teachers with the goal of generating general impressions. After considering the data, we drew out general impressions that are shared in this manuscript.

Results

Overall, teachers provided more instructional opportunities intended to engage students in the SMPs. Figure 1 shows the frequency of instructional opportunities for each SMP during the pre- and post-PD instructional lesson. A paired samples t-test demonstrated that the overall growth from pre- to post-PD was statistically significant, $t(35) = 12.058$, $p <. 001$, [2.50, 3.51]. The instructional average was 0.94 SMPs indicators in total during pre-PD instruction (SD = .71). Put another way, we found approximately one out of a possible 23 indicators for the SMPs during pre-PD instruction. The post-PD instructional average was nearly three times greater, 3.94 (SD = 1.45). This suggests that we found roughly four unique indicators of teachers' promotion of the SMPs during their post-PD instruction. A closer look into these data indicates that teachers seemed to focus their instruction on promoting some SMPs more than others.

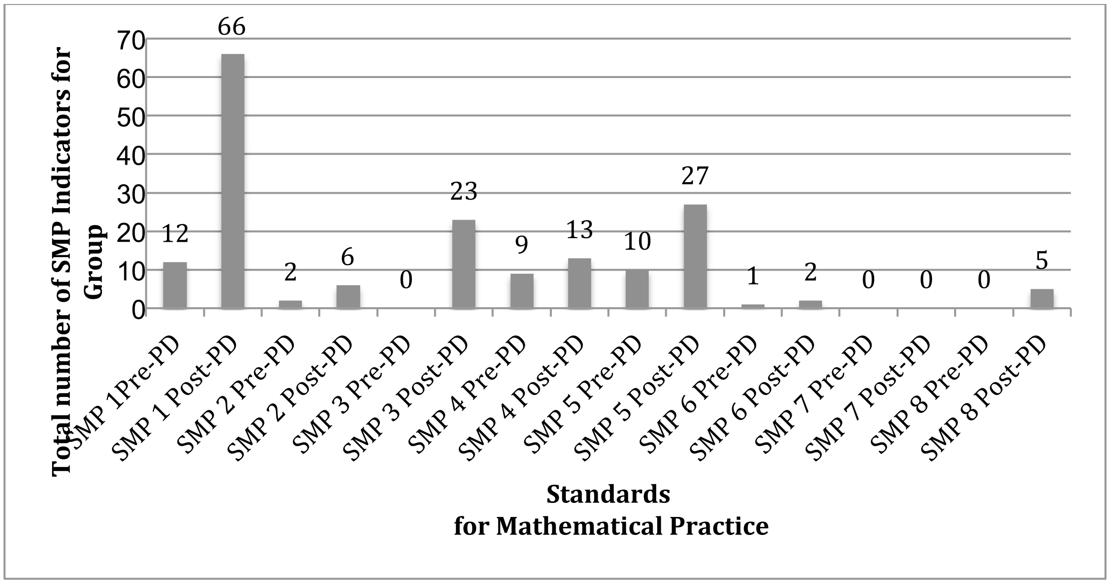

Figure 1. Frequency totals of observed SMPs indicators in pre- and post-PD instruction

We conducted chi-squared tests for group results related to each SMP, correcting for inflated error rates. The goal was to examine whether there were statistically significant differences between number of indicators within a SMP during pre-PD and post-PD instruction. Results are shown in table 2. Our results indicated that teachers' growth was statistically significant in four SMPs: SMP 1, SMP 3, SMP 4, and SMP 5.

Table 1. *Descriptive statistics and chi square results for SMP indicators of Pre-PD and post-PD*

	Pre-PD		Post-PD			
SMP	M	SD	M	SD	X^2	p*
1	0.33	0.48	1.83	0.84	60.92	<.001
2	0.56	0.23	0.17	0.56	4.24	0.06
3	0	0	0.64	0.59	29.65	<.001
4	0.25	0.44	0.36	0.49	15.86	<,001
5	0.28	0.45	0.75	0.6	36	<.001
6	0.03	0.17	0.06	0.23	2.06	0.08
7	0	0	0	0	-	-
8	0	0	0.14	0.42	4.24	0.06

* one-tailed interpretations

Taken collectively, these quantitative findings suggest that on average, teachers provided more opportunities for students to engage in the SMPs after the PD. Looking specifically at each teacher revealed that every teacher provided more opportunities to engage in the SMPs. We sought to qualitatively understand these changes with respect to the SMPs and three PD factors: learning environment, mathematical task, and discourse. Due to the brevity of this proceedings manuscript, we are only able to provide qualitative description of one teacher's instructional changes.

We noticed that instructional opportunities were clearly influenced by the implementation of their choice of task, changes in learning environment, and ways discourse was promoted. For example, a second-grade teacher's pre-PD instruction focused on guiding students through the definitions of a fraction in the context of exercise-laden teaching. Students were seated in rows and asked to follow her model of using pattern blocks to represent benchmark fractions. Then, students watched a video stemming from her textbook showing exactly the same activity as her students completed just minutes ago. Finally, students worked on a series of exercises without using pattern blocks. Students spoke only when the teacher asked a question. This directed instruction approach stands in stark contrast to her post-PD instruction.

The post-PD warm-up task was to determine how many letters there were in sum of the first names of the class. Students were seated in small groups and had access to a variety of manipulatives on their desks. The teacher encouraged several students to share how they counted the letters. After the warm-up task, she asked them to determine the number of legs in the classroom. The teacher utilized a think-pair-share approach with this task. Students used an initial representation (e.g., symbolic, graphical, verbal, and/or concrete) to solve this task and the teacher monitored students' work. She reminded students to explain what they were doing on their papers and to be prepared to justify why their approach is effective and efficient. As students finished working with an initial representation, she asked them to employ another viable representation to solve the problem. Finally, students shared how they solved the problem using multiple representations and then justified their strategy to a partner and then the class. Students also responded to questions from the teacher but the flow of discourse included multiple student-to-student interactions as well. It was apparent how the teacher provided an opportunity for her students to decontextualize the mathematical elements from the task and later contextualize the mathematical symbols with the referents in the problem. Through these instructional changes and ones like it, our sample of teachers provided greater instructional opportunities for students to engage in the mathematical practices.

Implications

From this study, we learned that teachers overwhelmingly engaged in greater opportunities related to the SMPs after the PD than before it. These changes are associated with modifications to the learning environment, mathematical task, and/or ways that the teacher initiated and sustained mathematical discourse. For example, the second-grade teacher's post-PD changes are tied to all three instructional aspects. These changes led to greater opportunities to foster students' engagement in the SMPs. While it is beyond the scope of the research to link one aspect of the PD with these changes, the results more broadly suggest that yearlong PD focusing on the CCSSM and our three central instructional aspects led to changes in the way these K-10 teachers designed and implemented mathematics instruction. The SMPs do not dictate curriculum or teaching but they do provide ideas for mathematically engaging students in classroom instruction. Sustained PD of a year or longer may help mathematics teachers at all grade levels make sense of mathematics instruction that supports students' appropriate mathematical behaviors. Results from this study support the prior literature suggesting that yearlong PD, which adheres to what works for designing and implementing effective PD (NCTM, 2007), tends to lead to instructional changes that promote improved opportunities to learn.

Limitations

Qualitative approaches allow researchers to draw on their lenses and frames of reference to make sense of experiences in the world. The results offered here are not generalizable to all teachers and are particular to this set of teachers. Our sample also limits some of the findings. That is, teachers volunteered to participate in the PD and those who are less motivated to complete yearlong PD may have different outcomes making instructional changes. Furthermore, teachers differing in some way from our sample in terms of years of experience, school district location, or other aspects might lead to other findings. A third limitation was that the pre-PD video was done after nine hours of Common Core PD. Thus, any growth in teachers' promotion of the SMPs is limited because they experienced some PD prior to their pre-PD instructional video.

Conclusion

The third limitation provides an important finding about the importance of our yearlong Common Core PD program. Teachers had another 78 hours of PD following their pre-PD videos, which is a strong indication of the impact that sustained PD has on teachers' instructional outcomes. That is, teachers provided limited opportunities for students to engage in the SMPs after nine hours of PD, yet improved greatly after more time to consider their PD experiences and translate them into pedagogical instantiations to promote the SMPs. The evidence found in this study suggests that K-10 teachers benefitted from reflecting and working to implement the CCSSM through three instructional areas: learning environment, mathematical task, and mathematical discourse.

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PROBLEM SOLVING IN PRESCHOOL: ONE PROGRAM'S ALIGNMENT TO NAEYC AND NCTM

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This article presents a portion of a larger study (Johnston, 2010) that investigated the alignment of one preschool program to recommendations outlined by NAEYC and NCTM (2010). Data and findings relate to how the teachers (N=6) and the curriculum materials provided preschool children opportunities to develop mathematical problem solving skills. Analysis of the data indicated few opportunities to engage in problem-solving situations. Results from this research help to provide the foundation for future investigations of how teachers of young children follow NAEYC and NCTM's (2010) recommendations.

Research supports the benefits of early learning opportunities (Barnett, Lamy, & Jung, 2005; Ou & Reynolds, 2006). Early mathematics experiences influence mathematical outcomes later in school (Lopez, Gallimore, Garnier, & Reese, 2007) and promote school readiness skills in mathematics (Gormley, Gayer, Phillips, & Dawson, 2005). Teachers' misconceptions of appropriate high-quality mathematical learning opportunities for young children may hinder their ability to incorporate new mathematical standards (Lee & Ginsburg, 2009). The National Association for the Education of Young Children (NAEYC) and National Council of Teachers of Mathematics (NCTM) have created specific recommendations related to high quality mathematics instruction in the early childhood classroom (NAEYC & NCTM, 2010; NCTM, 2000). Variance is present in programs for young children, so it is important to insure that these early learning opportunities are high quality (Varol & Farran, 2006). Once these recommendations or standards are disseminated to the public it is necessary for researchers to determine whether teachers understand these guidelines and how they are being used in the classroom.

Related Literature

To support NCTM's inclusion of prekindergarten in its *Principles and Standards for School Mathematics* (PSSM) (2000), NCTM and NAEYC published a joint position statement in 2002 (updated in 2010) titled "Early Childhood Mathematics: Promoting Good Beginnings". In this document, NAEYC and NCTM (2010) outlined 10 recommendations to help teachers provide high-quality mathematics instruction for children ages 3 to 6 years old. Suggestions include the importance of play in mathematical learning, supporting children's natural interest in mathematics, and including connections to children's everyday routines as well as other content areas. The emphasis of this article is on the ideas from Recommendation 4, which focuses on problem solving in the mathematics classroom, providing children with the opportunity to participate in mathematical conversations and representing mathematics in a variety of ways.

Research did not specifically focus on preschoolers as problem solvers despite the emphasis in NAEYC & NCTM (2010) recommendations. However, research supported kindergarteners as successful problem solvers (Carpenter, Ansell, Franke, Fennema, & Weisbeck, 1993) and recognized first grader's ability to develop a variety of solution strategies (De Corte & Verschaffel, 1987). Providing primary children with frequent opportunities to engage in meaningful problem solving increased their ability to complete these activities successfully (Fennema et al., 1996). In addition, kindergartners had opportunities to practice computational skills without the need for drill and practice activities. This type of learning environment increased teachers' understanding of children's problem solving strategies (Warfield, 2001). Salient characteristics of a classroom environment that supported primary children's emerging problem solving abilities included: having opportunities to participate in conversations related to problem solving strategies, engaging in successful problem solving opportunities, and having teachers who accept varied solution methods (Fennema et al., 1996).

Methodology

The purpose of this study was to determine the extent to which a preschool program followed the recommendations outlined in NAEYC and NCTM's position statement "Early Childhood Mathematics: Promoting Good Beginnings" (2010). This paper will describe the findings related to problem solving opportunities in this preschool program (Recommendation 4).

This study took place in a large city in northern Texas with one NAEYC accredited preschool program consisting of 24 classrooms at 6 different locations. Thirteen of the classrooms were part of the Head Start program and the other classrooms were tuition based with a sliding scale according to the income of the family. With a student population of 89% Hispanic, one of the primary goals of this program was to provide a language-rich environment to help all children gain a strong foundation in the English language. Both the Head Start classrooms and the tuition-based classrooms followed the same curriculum for children ages 2 ½ to 5 years.

The sample for this study included 3 of the 6 sites from this preschool program. Two teachers from each of the 3 sites were randomly selected to participate in this study ($N = 6$). All 6 participants in this study were female and Hispanic. Table 1 provides additional demographic data about the participants.

Table 1. *Teachers' Educational Background and Teaching Experience*

	Teacher					
Characteristic	A	B	C	D	E	F
Education						
Highest degree[a]	BA	CDA+	CDA+	CDA	CDA+	AA+
Highest EC degree	AA	CDA	CDA	CDA	CDA	AA
Experience						
Years at program	6	5	13	5	8	8
Total years teaching	6	5	13	8	8	15

Note. CDA: Child Development Associate; AA: Associate degree; BA: Bachelor's degree
[a] + indicates that teacher is taking courses to obtain a higher degree

Two research questions guided this study--

1. To what extent do the preschool teachers' instructional practices follow the recommendations outlined in NAEYC and NCTM's position statement "Early Childhood Mathematics: Promoting Good Beginnings" (2002)?
2. To what extent does the preschool program's curriculum align with the recommendations outlined in NAEYC and NCTM's position statement "Early Childhood Mathematics: Promoting Good Beginnings" (2002)?

Data collection included the use of the Classroom Observation of Early Mathematics—Environment and Teaching (COEMET) (Sarama & Clements, 2007). Recommended by Kilday and Kinzie (2009) for the preschool classroom, the COEMET provides observers with a framework for observations related to mathematics. The specific math activity (SMA) section of this instrument includes two different types of math activities. A full SMA consists of any mathematics activity where the teacher is involved in the learning process. For instance, Item 13 states "The pace of the activity was appropriate for the developmental levels/needs of the children and the purposes of the activity." For this item, the observer rates the pace of the SMA using the following scale: *strongly disagree*, *disagree*, *neutral/not applicable*, *agree*, *and strongly agree*. The SMA component contains 22 items. However if "the activity called for no extensive discussion of concepts or strategies" then the observer only completes the first 8 items (Sarama & Clements, 2007). In addition, there is a place for the observer to include a short description of the math activity. A mini SMA (mSMA) is a classroom activity that is student directed or an activity where the teacher does not focus on the mathematics components inherent to the exercise. For example, finger plays, rote calendar activities, or children's use of mathematical materials without instructions from the teacher qualify for this type of SMA.

Using the COEMET each teacher was observed 6 times over the course of a 2-month period during the spring. In addition, the observer wrote field notes about each observation. To give support and extend the information obtained from the COEMET, each teacher was interviewed 3 times. The interview questions either aligned with one of the recommendations described in NAEYC and NCTM's (2002) position statement or provided information about the teachers' educational background, teaching experience, or participation in professional organizations.

To fully understand the mathematics instruction at this preschool program, weekly lesson plans (6 plans from each teacher) and all curriculum materials related to mathematics were reviewed and coded for alignment with NAEYC and NCTM's recommendations for problem solving opportunities and mathematical discourse.

Findings

Many of the SMAs did not use the full protocol. The total number of SMAs observed during this study for all the teachers was 152. Only 14 of these math activities warranted the use of the extended protocol or 9.21%. Although nine items from the COEMET related to problem solving, only one item (12) will be discussed since items 17, 19-23 and 25-26 are from the extended protocol. This lack of data indicated that many of the SMAs did not address the ideas presented in this recommendation.

Item 12 rated whether a teacher started a SMA with an engaging mathematical question or idea. Ratings ranged from 1.92 to 2.94 ($M = 2.40$). These averages indicate that the majority of SMAs did not begin by focusing children's mathematical thinking or that the teachers did not provide these opportunities on a consistent basis. Examples that provide evidence of the teachers' ability to engage students in mathematical thinking included asking the children questions such as:

"How many words are in this sentence?"

"Whose castle is taller?"

"How many bugs do you think are in this box?"

"How many cups does it take to fill this container?"

Often, teachers did not offer such opportunities. For instance, sometimes teachers included center activities but did not introduce these new learning opportunities to the class, or teachers asked the children to copy shapes in their journals without any discussion about these mathematical figures.

Children did not have many opportunities to develop the process skills described in this recommendation as indicated with the lack of data from the second part of the COEMET. Sometimes teachers did provide short opportunities, usually in a conversation with an individual child, to develop problem-solving skills. For example, Teacher E asked a child to show 20 using his fingers. The child told the teacher that he does not have enough fingers. The teacher asked him, "How many fingers do you have?" The child counted his fingers and told the teacher 10. Then the teacher asked, "What can we do?" The child was unsure. The teacher then showed how the child could count beyond 10 by using his fingers.

Two major themes emerged from the interview transcripts related to the ideas presented in Recommendation 4. First, the teachers reflected about their approaches to teaching problem solving to their students. Table 2 lists the different viewpoints teachers indicated when discussing these activities.

Table 2. *Teachers' Comments About Problem Solving Activities*

Problem Solving Viewpoint	Number of Teachers
Need to incorporate more opportunities	2
Include problems related to estimation	3
Children are too young	1
Do not focus on problem solving	3
Use simple problems	1

Teachers A and B indicated that problem solving is something that they do not incorporate often during math although both teachers provided examples of problem solving when discussing other components of their math program. For example, when talking about centers, Teacher A stated that she might ask children various math questions in the block area such as "Can you build a tower with only 15 blocks?" Three teachers talked about providing children with problem solving opportunities by asking them to estimate different situations. In addition, Teacher F described how the children solve other simple problems. She provided the following examples:

> For example, one day a jar full of flowers and we asked them to estimate how many flowers in the jar and we did the same using frogs and then we count them together. We

found out who had the closest number. We also told them, "We have five insects, if the frog eats three, how many are left?"

Teacher D was the only teacher who believed that the children were not able to understand problem-solving situations.

None of the activities listed in the lesson plans indicated the use of the problem solving skills described in this recommendation, however the curriculum materials provided some examples. For example, several of the activities included possible questions to ask children while completing the lesson. These questions varied from basic ones with only one correct answer to more complex ones that would require students to use higher order thinking skills. *Mathematical Discoveries for Young Children* (1992) provided the most support to teachers in this area. Simple questions included:

- "Are there more children, or are there more chairs?" (p. 35)
- "Are both trains the same height?" (p. 57)
- "What is first in your line?" (p. 80)

Questions requiring use of reasoning or other process skills included:

- "Why does this belong? Do you have a rule for putting these things together?" (p. 6)
- "How do you know?" (p. 33)
- "Did anyone arrange their counters in another way?" (p. 52)
- "Tell me something you discovered about these shapes" (p. 67).

Some of the activities presented in the curriculum provided rich learning experiences that allowed children to explore mathematics through developing a solution to a problem. One example was from the *Building Language for Literacy* (2000) resource. In this activity children explored building tall buildings. During these explorations, the children tried to answer the following questions:

- Do some blocks work better at the top or the bottom of your building?
- How tall does the building get before it has trouble staying up?
- What can you do to keep it from falling over?" (BLL, p. 65).

Although some learning experiences incorporated problem solving and reasoning opportunities, many activities did not require children to use these processes. Often tasks included in the curriculum presented students with a skill based learning experience.

Discussion

When observing a SMA, the observer only completes part of this observation tool if "the activity called for no extensive discussion of concepts or strategies" (Sarama & Clements, 2007, COEMET SMA page 1). This section of the protocol was left out of the analysis because few of the SMAs required the completion of the full instrument. Many of the items addressed in this section related to the process skills addressed in Recommendation 4 (NAEYC & NCTM, 2002). This lack of data indicated that teachers did not provide many opportunities for children to engage in meaningful problem solving opportunities in which they had the opportunity to discuss possible solution strategies. Also, many of the math concepts children confronted during these observations did not require teachers to engage children in comprehensive discussions of these ideas.

There are several possible reasons for this situation. First, teachers indicated that as the school year progresses, more of the math content presented to the students is review from earlier in the year. Since these observations occurred during the last 2 months of the school year, many of the math activities may have been review items. Teachers may not have seen the need to engage children in long discussions about these ideas because they had been presented earlier in the year. Second, the curriculum materials did not include a plethora of problem solving activities. In addition, these activities did not always include suggestions of how to develop mathematical conversations with children about their strategies or approaches to solving these situations. This lack of support from the curriculum may have hindered the teachers' ability to provide these learning experiences for children. Lastly, teachers own experiences with problem solving may influence how they incorporate these ideas into their classroom. Teachers uncomfortable with solving problems or with their mathematical ability may struggle to engage children in discussion about the strategies they used or their reasoning for approach a problem in a certain way.

High-quality professional development experiences could provide teachers in this program the support they need to include more problem solving opportunities into the learning activities. These positive early learning experiences may also help to provide the necessary foundation for success in mathematics, which in turn will help create children with more positive mathematical views.

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ELEMENTARY MATHEMATICS TEACHER BELIEFS

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The purpose of this study was to understand teacher beliefs about teaching mathematics over the course of an elementary mathematics teaching methods course. The participants came from three groups of in-service and preservice teachers in master's degree programs at a university in New York: New York City Teaching Fellows, Teacher Education Assessment and Management program, and traditional preservice teachers. Findings revealed an increase in positive beliefs about teaching mathematics over the semester, but there were no significant differences in participants' beliefs among the three programs.

The purpose of this study was to understand teacher beliefs over the course of an elementary mathematics teaching methods course that emphasized problem-solving and constructivism for teachers. The participants in the study came from three unique groups of in-service and preservice teachers in master's degree programs at a medium-size university in New York: New York City Teaching Fellows (NYCTF) program, Teacher Education Assessment and Management (TEAM) program, and traditional preservice teachers enrolled in a graduate program at the university. All three programs were two-year graduate programs designed to prepare teachers for work in urban schools with certification in childhood and special education.

The NYCTF program is an alternative certification program developed in 2000 by the New Teacher Project and the New York City Department of Education (Boyd, Lankford, Loeb, Rockoff, & Wyckoff, 2007). NYCTF's goal was to bring career changers into education to fill the large teacher shortages in New York public schools. The TEAM program is a partnership between the TEAM organization and the partnering university. TEAM is an organization that facilitates partnerships with universities for its student members, who receive a tuition discount due to the negotiated tuition rate (TEAM, 2012). Cohorts generally consist of 12 to 20 Orthodox Jewish teachers. Traditional preservice teachers were enrolled in the university's graduate program, which required extensive fieldwork. Participants in the program were required to participate in 10 hours of fieldwork for each three-credit class in which they were enrolled.

Theoretical Framework

This study is grounded in sociocultural theory (Vygotsky, 1987), which proposes individual learning is framed by experiences in learning socially among others. In the classroom context this interaction occurs between instructor and student and also among the students. This methods course was framed by teaching mathematics from a problem-solving perspective, as proposed by the National Council of Supervisors of Mathematics (NCSM) (1978) and National Council of Teachers of Mathematics (NCTM) (2000). NCTM (2000) said, “Problem solving is not only a goal of learning mathematics but also a major means of doing so” (p. 52). Mathematics should be taught in a manner so that students are solving unfamiliar problems using their previously acquired knowledge, skills, and understanding to satisfy the demands of unfamiliar situations (Krulik & Rudnick, 1989).

Teachers’ beliefs about teaching mathematics are important for teacher quality (Charalambous, Panaoura, & Philippou, 2009), and their beliefs influence the manner in which they teach, the content they teach, and teacher-student interactions (Beswick, 2012; Hart, 2002). Teacher beliefs can also impact student performance (Cousins-Cooper, 2000; Ladson-Billings, 2009; Leonard & Evans, 2008). Lortie (1975) claimed teacher beliefs are developed at an early age before they enter teacher preparation. Although the literature indicates the stability of beliefs over time (Connor, Edinfield, Gleason, Ersoz, 2012; Reeder, Utley, & Cassel, 2009) and beliefs are found to be difficult to change, it is possible that beliefs are malleable and can be changed through experience in teacher preparation (Pajares, 1992). Moreover, Pajares (1992) said, “all teachers hold beliefs, however defined and labeled, about their work, their students, their subject matter, and their roles and responsibilities” (p. 314). In this study, teacher beliefs can be defined as the combination of beliefs about teaching mathematics, the nature of mathematical knowledge, mathematical confidence and efficacy, and the belief in the degree of student-centered and inquiry-based instruction.

Research Questions

The following research questions are important to answer in order to determine significant differences in beliefs about teaching mathematics before and after the participants had taken a reformed-based mathematics methods course. The author had particular interest in determining differences among the programs given that the participants from each program come from different population groups. This is important because a) the author wanted to determine if beliefs changed over the course of the semester and b) the author wanted to know if beliefs varied among the different preparation programs. The author suspected there could be differences between the programs given the different populations for each group. Finally, the author wanted to know specifically what those beliefs were. The following are the research questions for this study.

1. Were there significant differences in beliefs about teaching mathematics before taking and after having taken a reformed-based mathematics methods course?
2. Were there significant differences in beliefs about teaching mathematics among the NYCTF, TEAM, and traditionally prepared teachers?
3. What were teacher beliefs about teaching mathematics?

Methodology

The methodology for this study was quantitative. The sample consisted of 115 preservice and in-service teachers. NYCTF teachers were all in-service teachers for this study, and TEAM and traditional teachers were primarily preservice teachers for this study, with several TEAM participants teaching in Yeshiva and Hebrew Academies. There were 84 NYCTF teachers, 16 TEAM teachers, and 15 traditional preservice teachers. Participants were enrolled in an inquiry- and reformed-based elementary mathematics methods course in the 2011/2012 academic year that involved both pedagogical and content instruction and was aligned with the NCTM *Principles and Standards for School Mathematics* (2000). The course was designed to prepare teachers to teach mathematics to elementary school students. It focused on elementary school mathematics content to an extent, but in the context of understanding the concepts behind the basics and number sense. The primary focus of the course was on how to teach mathematics for student understanding.

Teachers were given the validated Mathematics Beliefs Instrument (MBI) at the beginning and end of the semester, which was created by Hart (2002) and measured participants' beliefs about teaching mathematics. The MBI is a 30-item 5-point Likert scale instrument that solicits participant beliefs about reformed-based methods of mathematics instruction, such as problem-solving, conceptual understanding, and student-centered teaching that includes active student participation. Higher scores indicate more positive beliefs on the 5-point Likert scale, and thus average scores used for statistical analyses are averages of overall Likert scores with high scores indicating more positive beliefs.

Results

A paired-samples *t*-test was conducted to answer research question one in order to determine significant differences in the MBI scores over the course of the semester. While it was not expected that positive beliefs would decline over the semester, a two-tailed test was conducted. A statistically significant difference was found at the 0.05 level between the pretest (M = 3.56, SD = 0.333) and the posttest (M = 3.66, SD = 0.350) with $t(114) = -3.970$, $p = 0.000$, $d = 0.29$, two-tailed. This indicated an increase in positive beliefs about teaching mathematics with a small effect size.

One-way ANOVA was conducted to answer research question two in order to determine significant differences in MBI scores among NYCTF, TEAM, and traditional teachers. No statistically significant differences were found between NYCTF, TEAM, and traditional teachers for the pre- and post-tests. It should be noted that there is limitation in conducting an ANOVA for these three groups given the differences in available sample sizes.

Descriptive statistics were used to answer research question three. Results indicated teachers felt most positively about the study of mathematics including opportunities of using mathematics in other curriculum areas; mathematics must be an active process; and mathematics can be thought of as a language that must be meaningful, if students are to communicate and apply mathematics productively. Teachers felt positively about beliefs generally considered negative by reform-oriented mathematics educators, such as emphasizing clue words (key words) to determine which operation to use in problem-solving; some people being good at mathematics and some people not being good at mathematics; and mathematics as a process in which students absorb information, storing it in easily retrievable fragments as a result of repeated practice and reinforcement.

Discussion

Findings revealed an increase in positive beliefs about teaching mathematics, but there were no differences in participants' beliefs among the three programs. Teacher beliefs included using mathematics in other curriculum areas, mathematics as an active process, and the communication aspects of mathematics as a language.

It was found teachers felt most positively about the study of mathematics including opportunities of using mathematics in other curriculum areas; mathematics must be an active process; and mathematics can be thought of as a language that must be meaningful, if students are to communicate and apply mathematics productively. While it is important teacher educators continue to encourage teachers in these areas, it is more important that teacher educators work with teachers in areas in which they felt less positively. Teachers believed in emphasizing clue words (key words) to determine which operation to use in problem-solving; some people being good at mathematics and some people not being good at mathematics; and mathematics as a process in which students absorb information, storing it in easily retrievable fragments as a result of repeated practice and reinforcement.

The emphasis of clue words for finding solutions to word problems does not lead to true conceptual understanding that students need to solve unfamiliar problems, which is the primary component of authentic mathematical problem-solving. Teachers who emphasize clue words are assisting students to rely on procedures demonstrated by the teacher without actual student understanding. Teacher educators must help their preservice and in-service teachers foster an environment of true understanding by instead assisting their students to use their previously obtained skills, knowledge, and understanding to satisfy the demands of an unfamiliar situation (Krulik & Rudnick, 1989). This can be modeled through problem-solving in teacher preparation classes using multiple types of problems and unfamiliar situations for the teachers, which they can bring into the classroom.

A major limitation for this study is the generalizability of the findings. The sample in this study was restricted to one university in New York with unbalanced sample sizes for the three groups studied. The samples represented convenience samples due to availability. Further study should be conducted on teacher beliefs with larger samples from these groups. This study should be considered exploratory.

Teacher beliefs are an important component of teacher quality, and teacher educators can influence teachers' beliefs to help them become better teachers, which leads to higher student achievement and success. It is not enough for teacher educators to focus only on content knowledge and pedagogical skills, although these are certainly important variables for student achievement and success; there must also be emphasis on understanding teacher beliefs, and challenging and shaping those beliefs, which will consequently lead to higher student achievement and success.

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LEARNING TO LISTEN: WHAT A PRE-SERVICE TEACHER CAN LEARN FROM AN INTERVIEW

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Understanding and making sense of students' mathematics learning is an essential part of becoming an exemplary mathematics teacher. In this initial study, pre-service teachers were asked to conduct clinical interviews with P-8 students. Listening to students rather than teaching or correcting, and making sense of a students' mathematics was the focus of this assignment. Pre-service teachers found it difficult to listen to students as well as to make sense of a student's mathematics.

Standardized testing is a predominate method for prospective and practicing teachers to make sense of students' mathematics. In spite of their extensive use, these tests may not provide a rich view of a student's mathematical learning and ideas. A clinical interview, however, has the potential to provide prospective or practicing teacher an opportunity to make sense of a student's mathematical world or a small glimpse into their mathematical understanding and sense making.

Theoretical Framework and Related Literature

The NCTM *Principles to Actions: Ensuring Mathematical Success for All* (2014) addresses several realities, two of which are:

- Too much focus is on learning procedures without any connection to meaning, understanding, or the applications that require these procedures.
- Too much weight is placed on results from assessments – particularly large-scale, high-stakes assessments – that emphasize skills and fact recall and fail to give sufficient attention to problem solving and reasoning. (p. 3)

This leads to many students with weak mathematics understanding. In describing effective teaching and learning, *Principles to Actions* (NCTM, 2014) goes on to state:

> An excellent mathematics program requires effective teaching that engages students in meaningful learning through individual and collaborative experiences that promote their ability to make sense of mathematical ideas and reason mathematically. (p. 7)

As the goal of mathematics teachers should be to promote mathematics learning and sense making, it is imperative that prospective teachers have opportunities to reflect upon what it means to learn and do mathematics and what sense students make of their mathematics classroom experiences. Traditional assessments tend to inform teachers of what they already know about a student whereas non-traditional assessments, such as clinical interviews, can "inform teachers of students' understanding and suggest instructional modification" (NCTM, 2014, p.91). As Ginsburg (1997) notes the clinical interview provides an opportunity to enter the child's mind:

> The clinical interview can help you understand how children construct their personal worlds, how they think, how their cognitive processes (at least some of them) operate, how their minds function. I believe that the clinical interview can make important contributions both to basic research and to applications in the clinic, the school, and elsewhere too, like the courts and the physician's office. (p. 28)
>
> Because people's minds are so extraordinarily complex, we must expand our methodology to include the *deliberately nonstandardized* approach of the clinical interview. (p.29)

This is more than a surface level one-dimensional look at learning and sense making. Because students are unique, we must treat each one with respect and fairness and be flexible in our assessment. By listening to students, prospective teachers have an opportunity to not only assess a student's mathematics but to begin a construction of how students come to know mathematics and deepen their understanding of students' mathematics learning (Ellemore-Collins & Wright, 2008).

Methodology

Twenty-two participants were instructed to conduct a clinical interview with a P-8 student of their choice. These participants were prospective teachers who were enrolled in a mathematics course – Mathematical Reasoning – the last of four mathematics courses that are specific to early childhood majors. During this semester all of the early childhood majors were also involved in a clinical experience at various local elementary schools three days each week. The preservice students were in various levels for their practicum experience from pre-kindergarten through fifth grade. There were also two middle grades education majors that were enrolled in this mathematics course.

Preservice students were first asked to read various articles discussing clinical interviews. In addition, they were given suggested questions for various levels of students. The texts assigned for reading included "Assessing for Learning: the Interview Method" (Labinowicz, 1987) and "Informing Learning through the Clinical Interview" (Long & Ben-Hur, 1991).

Suggested clinical interview questions with recommended grade levels were given that included tasks such as screened tasks, mental mathematics problems, problem solving using various operations, spatial tasks, and conservation of number tasks (G. Wheatley, personal communication, 1993). Class time was taken to discuss conducting clinical interviews with the focus being on making sense of a student's mathematics. Preservice students were instructed to *listen* rather than teach, as well as to be nonjudgmental in their communication with the student. They were also advised to record their interview when possible.

Preservice students were then asked to write up the results of their clinical interview, to include pertinent information about the student, questions asked, and responses given. They were to construct an explanation of the interviewed P-8 student's activity and responses and infer their student's reasoning. In addition to the written report, each prospective teacher had an individual debriefing session with their mathematics instructor to examine their first clinical interview experience and discuss follow-up questions and strategies for a future clinical interview. Some of these interviews were audiotaped. The preservice teacher participants also responded to written open-ended questions after completing the interviews and analysis:

1) What did you learn from this experience?
2) What was the most difficult part of this assignment?
3) What would you to the same or differently?
4) Is this something that you could use formally or informally as a practicing teacher?

The explanations that were constructed to examine the prospective teachers' interviews were from an interactionist perspective (Blumer, 1969; Bauersfeld, 1988, 1991). The participants were seen as acting according to their meanings of mathematics and mathematics learning as they interacted with their students. In attempting to make sense of their clinical interview, it was important to listen to the different participants and provide opportunities for each to elaborate upon their experiences. By examining their written reports, interviewing each prospective teacher about their clinical interview experience, and looking at their written responses to the open ended questions, there were opportunities for making sense of and analyzing this interviewing experience for these future teachers by looking for recurrent patterns and presenting sample episodes to illustrate general claims and assertions (Atkinson, Delamont, & Hammersley, 1988; Taylor & Boydan, 1984; Voigt, 1992).

Findings and Discussion

In the interview, the prospective teachers shared that their mathematics experiences had occurred in predominantly traditional mathematics classroom where the teacher demonstrates and the student practices. The tasks suggested for this clinical interview could be classified as performance questions as the interviewees were typically asked how they solved the task and why they chose to solve it in their particular way (Zazkis, R. & Hannan, 1999; Heng, M. A. & Shudarshan, 2013). Because of their previous experiences in mathematics classrooms and with mathematics assessment, these prospective teachers found it difficult to construct an explanation of their student's activity and infer the student's reasoning. One of the prospective teachers stated that "typing up a paper explaining everything she (the interviewee) did was a little difficult." Others stated:

> "I think the most difficult part of this assignment was analyzing the results and thinking about the student's mathematical reasoning skills. It was challenging to analyze what my student's responses meant in reference to mathematical concepts, reasoning, and skills. I feel like this is something I will improve on the more I teach and am familiar with the content."
>
> "I also learned that it is important to listen to their answers and question why those are the answers. It is one thing for a student to give an answer, but it is quite another to listen to their thought processes."

Assessing a traditional assessment with specific right/wrong responses was much easier for the participants, although there was no indication that a traditional assessment was a richer assessment instrument. Students typically described their interviewee's actions in a superficial way such as being able to complete a particular operation, for example a traditional addition, subtraction, multiplication, or division algorithm.

Most of the interviewers stated that changing roles was also difficult task for them. As these prospective teachers view their role of provider of answers or demonstrator of procedures and algorithms, they stated that listening as the students worked through a problem was difficult for them:

> "I think the most difficult part of this assignment was getting out of teacher mode! Every time she answered a question I saw it a teachable moment. I wanted to correct her and help guide her to the right answer."

"The most difficult part of this assignment for me was to be quiet. So many times I wanted to give the student an answer or ask if she wanted to reconsider her answer."

"I would say that the most difficult part of this assignment would be having to step out of the teacher role, to listen, and take anecdotal notes. I have been trained to correct, assess, and instruct at all times. Not being able to do any of these things made me step out of my comfort zone."

Changing roles and challenging the idea of what a teacher's role might be became a part of this process. Interviewers were perturbed about simply listening to learn from students. Not only does this point to their idea of what role a teacher should play, but it also points to their beliefs about how students learn and do mathematics. It also suggests that mathematics should be done quickly in a prescribed manner.

In addition to challenging the role of teacher and what it means to learn and do mathematics, the interviewers struggled with waiting on students to consider a task.

"The most difficult part of this assignment was knowing exactly how much time to spend on each problem without it being too much time. I could tell that my student was struggling, but I didn't want to stop her before she reasoned her way through."

Wait time was not easy for these prospective teachers. Watching students struggle was problematic to these interviewers.

Encouraging students to share their thinking was not as easy as the prospective teachers anticipated:

"I had chosen a student that I thought would communicate well; however, while doing the interview it was harder to get her to become clear in her explanations."

"…; it was kind of hard to really pull any deep explanations out of him. It could be because he felt uncomfortable with sharing his answers because it isn't anything that he's had to do before. There wasn't anything for him to write down (for me to try and make sense of as well). Having to read both his verbal and nonverbal communication cues and make judgments based upon those two factors were kind of difficult as well to me."

In discussing their clinical interviews, students discussed their struggle to encourage the interviewee to share and reflect upon their problem solving. Perhaps there were several factors at play here, one might be, as the student pointed out, the students had not been asked to do this before. In addition, reflection was not valued, but speed and accuracy were at the forefront of their mathematics classroom activities.

In spite of being given suggestions for tasks/problems, these prospective teachers were not sure about what questions might be appropriate for their students. Some were ill prepared with appropriate questions for some of the interviews. Some of the students quickly answered the questions asked of them, leaving the interviewer with little evidence about the student's mathematics thinking and sense making. They said in retrospect that they would have more possible questions/tasks for their students.

Many of the prospective teachers came to recognize that the students they interviewed viewed mathematics as a series of rules and procedures that were provided as they prepared for their next test or quiz. After interviewing a student that had been identified by her classroom teacher as "middle" performing, one prospective teacher noted that she had a better understanding as to why students identified as "middle" were "left behind" as they "only remember things just for the tests." She was surprised and disappointed at the student's lack of understanding. Many of the interviewees stated that their students were unable to "think outside the box" and indicated they were "stuck" on "algorithms and rules".

In the analysis of their initial clinical interview, the prospective teachers reflected upon their experience and noted that the interviewed students were seldom encouraged to be reflective and how difficult it would be if one had never been encouraged to do so:

> "I learned that my student has memorized mathematical operations and the conditions under which these operations are to be utilized. If he hears a specific phrase in a problem he automatically jumps to specific operation. I notice that he picks the numbers out of the problem and searches for clue words to tell him which operation he should be using. He doesn't seem to think about the reality of what the problem is asking. His solving methods are very robotic. He wasn't very driven to work through confusion and he did not persevere through problem solving."
>
> "By conducting this interview, I realized that students are taught a concept at an early age, but they aren't given an adequate explanation as to why things are the way they are until later on in their schooling. It's as if young children are to trust and believe what teachers tell them in their primary years … This is not always the best approach,

because if students actually have a chance to know why things are the way they are, then they may be more successful at figuring out what works for them."

While the interviewers describe a traditional mathematics classroom experience, it is encouraging to note that prospective teachers can reflect upon their experiences and call into question what experiences these students have had in mathematics classrooms.

This initial study provides evidence that prospective teachers can benefit from conducting clinical interviews with P-12 students. While their ability to analyze is in the beginning phases of reflecting upon students' mathematical sense making, they were challenged to consider mathematics learning and teaching. In future studies, students may be asked to videotape each interview to strengthen the process and help with analysis. In addition, viewing and discussing taped clinical interview before conducting one would also enhance the experience and analysis. The focus was to help prospective teachers to learn to listen as a way to assess student's mathematics and sense making so that they in turn can be negotiators of rich mathematics classrooms. Their reflections demonstrate that these clinical interviews have the potential to be beneficial in developing effective teachers.

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THE RELATIONSHIP BETWEEN TEACHER RELATED FACTORS AND MATHEMATICS TEACHERS' EDUCATIONAL BELIEFS ABOUT MATHEMATICS

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This study investigates the extent to which: (a) mathematics teachers' educational beliefs about mathematics change upon participation in professional development, and (b) teachers' educational background and teaching experience in mathematics contribute to their educational beliefs and to the change in these beliefs. Results showed that teachers significantly improved their educational beliefs about mathematics after professional development. Multiple regression analyses revealed that mathematics teaching experience predicted self-efficacy in teaching mathematics at program onset whereas mathematics college hours predicted the change in self-efficacy in teaching mathematics. The paper discusses the implication of findings for preparation and professional development of mathematics teachers.

At a time when certain measures of teacher quality are reduced to the growth of their students' learning, it is critical not to discount the educational beliefs associated with effective teaching. Several types of educational beliefs held by teachers have been identified as adaptive and associated with student success (e.g., Caprara, Barbaranelli, Steca, & Malone, 2006); however, little research has investigated the antecedents of these beliefs among practicing mathematics teachers (Stevens, Aguirre-Munoz, Harris, Higgins, & Liu, 2013). Moreover, research examining the extent to which professional development (PD) can promote these educational beliefs is scant. PD of teachers has been identified as one of the key factors of improving public education (Borko, 2004). Studies show that PD programs for teachers can improve not only teachers' knowledge and skills but also their beliefs, attitudes, and instructional practices with consideration of contextual factors such as school leadership and policy, curriculum, and characteristics of teachers and students (Desimone, 2009).

Theoretical Background

The focus of this paper was to examine antecedents of teachers' self-efficacy, internal locus of control, and epistemic beliefs. Teachers' self-efficacy may be defined as the extent to which teachers believe they can successfully execute teaching-related tasks within a particular context (Tschannen-Moran & Hoy, 2001). Initial work attempting to conceptualize and operationalize teachers' self-efficacy beliefs stemmed from a theory of locus of control (Rotter, 1966). Measures informed by this theory assessed how much control teachers felt they had over student outcomes regardless of external circumstances (e.g., outcome expectancy; Enochs, Smith, & Huinker, 2000). However, subsequent instruments developed to assess teachers' self-efficacy were more aligned with Bandura's (1986) definition of self-efficacy within a social-cognitive framework (Tschannen-Moran & Hoy, 2001). Thus, education researchers proposed that like other social-cognitive types of self-efficacy, teachers' self-efficacy is influenced by personal mastery experiences, vicarious experiences (observation of models), social persuasion, and physiological indicators (Schunk, Pintrich, & Meece, 2008). Proxies for these influential sources of teachers' self-efficacy examined in previous research include teaching experience, educational background in subject matter taught, and PD (Evans, 2014; Stevens et al., 2013; Wolters & Daugherty, 2007).

Locus of control beliefs are considered a motivational dimension within attribution theory that captures whether a person ascribes the causal factors of personal behavior and the behavior of others as being either internal or external (Weiner, 1992). Teachers' locus of control has been conceptualized as the extent to which teachers attribute student outcomes (i.e., achievement) to themselves or external factors (Rose & Medway, 1981). Prior findings indicate that an internal locus of control positively predicts job performance among teachers (Jeloudar & Lotfi-Goodarzi, 2012) and adaptive classroom behavior among students (Rose & Medway, 1981). Furthermore, research shows that an association exists between certification type (traditional vs. alternative) and teachers' beliefs about how much control they have over students' achievement-related outcomes (Evans, 2014).

Epistemic beliefs can be defined as an individual's beliefs about knowledge, which includes one's beliefs about where knowledge comes from, what the essence of knowledge is, and how one comes to know and justify beliefs. Educational psychology research has conceptualized and measured epistemic beliefs as residing across two ends of a spectrum. On one end, individuals believe that knowledge is fixed, simple, certain, objective, and comes from a person of authority. Muis (2004) classifies beliefs at this end of the spectrum as non-availing epistemic beliefs. Conversely, individuals classified as having availing epistemic beliefs view knowledge as evolving, complex, uncertain, subjective, and stemming from their own construction of knowledge. Availing epistemic beliefs are associated with positive motivational processes and academic achievement (Muis, 2004). Unfortunately, common characteristics of mathematics instruction (i.e., single formulaic approach to problem solving) hinder the development of more availing beliefs about knowledge. Therefore, in line with suggestions from previous researchers, it is imperative to examine antecedents of teachers' epistemic beliefs as these beliefs have been shown to influence instructional approaches, and in turn, students' own epistemic beliefs (Hofer, 2001; Muis, 2004).

The following research questions guided this study: (1) Did mathematics teachers' educational beliefs about mathematics change after a PD program? (2) What is the predictive value of background variables such as teaching experience, college mathematics hours, and teacher preparation route on teachers' beliefs about teaching and learning mathematics?

Method

This paper is a part of a larger study where we surveyed K-12 in-service teachers, who participated in three-week rigorous PD program. The PD aimed at improving teachers' pedagogical content knowledge (PCK)—the knowledge that they *"use in classrooms to produce instruction and student growth"* (Hill, Ball, & Schilling, 2008, p. 374). This included *knowledge of content and students*, *knowledge of content and teaching*, and *knowledge of curriculum* (Hill et al., 2008). We focused on two cohorts who participated in the study over the course of two summers (2013 & 2014) for this paper. Teachers either volunteered or were selected by their school administration to participate in the summer campus programs (SCP). The mathematical content focus of the 2013 SCP (first cohort) was: (a) numbers, operations, and quantitative reasoning; and (b) patterns, relationships, and algebraic reasoning. The content focus of the 2014 SCP (second cohort) was: (a) geometry, spatial sense, and measurement; and (b) data analysis, statistics, and probability. Both SCPs emphasized following research-based pedagogical constructs: active learning approach; motivation, applications, and problem-solving; and concept-based learning activities (e.g., Erickson, 2007; Pajares, & Graham, 1999). The total duration of each SCP was 72 contact hours (3 weeks; 4 days a week; and 6 hours a day).

In this study, 151 K-12 mathematics teachers (80 from cohort 1 and 71 from cohort 2) representing several urban school districts in the southwestern U.S. took pre- and post-surveys. Demographic breakdown of the teachers were 25% White, 39% African American, 26% Hispanic, 8% Asian, and 2% other. There were 118 female teachers (78%) and 33 male teachers (22%). Of all the teachers, 42 attended the elementary class (grades K-3); 35 attended the intermediate class (grades 4-6); 38 attended the middle school class (grades 7-8); and 36 attended the high school class (grades 9-12). On average, teachers took 21 college mathematics hours and had 3.5 years of mathematics teaching experience. In terms of preparation route, 42% had a traditional teacher preparation or master of arts in teaching, 50% went through alternative certification program, and 8% took other preparation routes (e.g., emergency, deficiency plan).

Participating teachers took a pre-survey two to three weeks prior to each SCP and a post-survey the last day of the SCPs. The survey items assessed teachers' self-efficacy in teaching mathematics, internal locus of control, and non-availing epistemic beliefs.

The survey consisted of several sections: 1) demographics, 2) teacher preparation background, and 3) Likert-scaled items adapted from previous scales (Mathematics Beliefs Instrument [Schoenfeld, 1989] and Mathematics Teaching Efficacy Belief Instrument [Enochs et al., 2000]) with adequate reliability and validity measuring the main constructs. All Likert-scaled items included in this study were rated on scales ranging from 1 (strongly disagree) to 5 (strongly agree), with higher scores indicating higher presence of the construct. Constructs and sample items are as follows: A measure of self-efficacy in teaching mathematics (e.g., "*I know the steps to teach mathematics concepts effectively*" [Enochs et al., 2000]), internal locus of control (e.g., "*Students' achievement in mathematics is directly related to their teacher's effectiveness in mathematics teaching*" [Enochs et al., 2000]), and non-availing epistemic beliefs about mathematics (e.g., "*Everything important about mathematics is already known by mathematicians*" [Schoenfeld, 1989]). Higher scores on the first two constructs are more adaptive; whereas, higher scores on the last construct—non-availing epistemic beliefs, are less adaptive.

The measures of self-efficacy in teaching mathematics (13 items), internal locus of control (8 items), and non-availing epistemic beliefs about mathematics (7 items) all had good reliabilities (with Cronbach's α's of 0.85, 0.75, and 0.72, respectively [Nunnally & Bernstein, 1994]). We calculated teachers' average scores (from 1 to 5) of all items as a composite score on each measure of educational beliefs for the pre- and post-surveys.

First, we compared the results of post-survey with that of pre-survey to explore the change in beliefs throughout PD to answer the first research question. Then, we investigated the predictive value of teacher-related background variables on the beliefs and the change in these beliefs from pre- to post-survey to answer the second research question.

Findings

We conducted paired-samples *t*-tests to investigate whether changes occurred in teachers' beliefs. Overall, the changes were significant ($p < .01$) with moderate effect sizes (0.47 to 0.64):

teachers' self-efficacy in teaching mathematics and internal locus of control increased (0.22 and 0.21 points, respectively) while their non-availing epistemic beliefs decreased (0.28 points; see Table 1). Then, we divided teachers into two groups (grades K-6 and grades 7-12) to see if changes in beliefs differed by grade level given that previous research indicates that elementary teachers have less mathematics background compared to higher grades. We conducted independent-samples *t*-tests to compare the two groups of teachers. Although there was not a significant difference ($p > .05$) on the pre-survey, K-6th grade teachers showed more change (growth) in their self-efficacy beliefs in teaching mathematics than 7th-12th grade mathematics teachers ($p < .01$ with an effect size of .55; see Table 2).

Table 1. *Paired-Samples t-test Results for Change in Measures of Teachers' Educational Beliefs*

	Paired differences (post-pre)				
Survey	N	Mean gain	S.D.	*t*-value	Cohen's *d*
Self-efficacy in teaching math	151	0.21905	0.42055	6.401*	.520
Internal locus of control	151	0.21109	0.45396	5.714*	.465
Non-availing epistemic beliefs	151	-0.28477	0.44523	-7.859*	.640

Notes. $^{*}p < .01$.

Table 2. *Independent-Samples t-test Results for Comparing Change in Beliefs between Grade Levels*

	N		Mean gain		S.D.			
Survey	K-6	7-12	K-6	7-12	K-6	7-12	*t*-value	Cohen's *d*
Δ Self-efficacy in teaching math	77	74	0.33	0.11	0.47	0.33	11.416*	.551
Δ Internal locus of control	77	74	0.22	0.21	0.42	0.49	0.018	-
Δ Non-availing epistemic beliefs	77	74	-0.34	-0.27	0.47	0.43	2.566	-

Notes. $^{*}p < .01$.

Table 3 shows means, standard deviations, and correlation coefficients among teachers' background variables, their scores on belief measures, and the change in belief scores. Results revealed that higher self-efficacy in teaching mathematics was associated with more availing epistemic beliefs ($r = -.20$, $p < .05$). Teachers' pre-survey scores in the three belief measures were negatively associated with growth in those measures (with *r*'s ranging from -.56 to -.33, $p < .01$). This was an expected result simply because teachers who began with higher scores had less room for improvement.

Table 3. *Means, Standard Deviations, and Pearson Correlations among the Main Variables*

Variable	*M*	*SD*	1	2	3	4	5	6	7	8	9
1.Years of math teaching	3.52	4.06	---								
2.Math college hours	21.6	15.8	.00	---							
3.Trad. teacher prep route	0.42	0.50	-.11	-.07	---						
4.Other prep route	0.08	0.27	.24	.30**	-.25**	---					
5.SE in teaching math	4.04	0.49	.21**	.07	.00	.12	---				
6.Internal locus of control	3.51	0.48	.07	-.06	-.15	.12	.11	---			
7.Epist. beliefs (non-avail.)	2.25	0.52	.06	-.04	-.02	.01	-.20*	-.08	---		
8.Δ SE in teaching math	0.22	0.42	-.08	-.19*	-.04	.00	-.56**	.03	.09	---	
9.Δ Internal locus of control	0.22	0.46	.00	.02	-.01	-.05	-.08	-.33**	.09	.21**	---
10. Δ Epist. beliefs (non-avail.)	-0.28	0.44	.07	.12	.02	.09	.05	-.12	-.41**	-.11	.01

Notes. $N = 148$; $^{*}p < .05$. $^{**}p < .01$.

Table 4 shows the results of six two-step hierarchical regression analyses conducted to predict the three belief measures and the change in these three measures. Variables associated with mathematics background were entered in the first step while preparation route variables were entered in the second as the first group of variables are specific to mathematics content.

Table 4. *Summary of Hierarchical Regression Analyses Predicting Educational Beliefs Among Mathematics Teachers*

Variable	Self-efficacy in teaching math[a]	Internal locus of control[b]	Non-availing epistemic beliefs[c]	Δ Self-efficacy in teaching math[d]	Δ Internal locus of control[e]	Δ Non-availing epistemic beliefs[f]
	β	β	β	β	β	β
Step 1 (math background)						
Years of math teaching	.20*	.03	.06	-.08	.02	.06
Math college hours	.05	-.10	-.04	-.21*	.04	.11
Step 2 (teacher prep route)						
Traditional	.05	-.12	-.01	-.05	-.02	.04
Other	.07	.11	.00	.08	-.08	.05

Notes. β indicates standardized regression coefficient. $N = 148$. $^{*}p < .05$. *ns* = not significant.
[a] Step 1/Step2: $R^2 = .05$, $p < .05$ / $\Delta R^2 = .01$, *ns*. [b] Step 1/Step2: $R^2 = .01$, *ns* / $\Delta R^2 = .01$, *ns*. [c] Step 1/Step2: $R^2 = .01$, *ns* / $\Delta R^2 = .00$, *ns*. [d] Step 1/Step2: $R^2 = .05$, $p < .05$ / $\Delta R^2 = .01$, $p > .01$. [e] Step 1/Step2: $R^2 = .00$, *ns* / $\Delta R^2 = .01$, *ns*. [f] Step 1/Step2: $R^2 = .01$, *ns* / $\Delta R^2 = .00$, *ns*.

In the regression predicting self-efficacy in teaching mathematics, after entering the two math background variables in Step 1, the model was statistically significant ($F(2, 145) = 3.84$, $p < .05$, $R^2 = 5\%$). Specifically, years of experience in mathematics teaching emerged as statistically significant ($\beta = .21$, $p < .05$). In the regression predicting change in mathematics teaching self-efficacy, Step 1 was statistically significant ($F(2, 145) = 3.08$, $p < .05$, $R^2 = 4\%$). The number of mathematics college hours earned was statistically significant ($\beta = -.21$, $p < .05$).

Discussion and Conclusions

This study informs us about the potential motivational benefits of a PD program focusing on PCK and expands our knowledge of antecedents associated with several educational beliefs among K-12 mathematics teachers. Findings suggest that PD aimed at enhancing PCK not only improves teachers' PCK but also promotes teachers' adaptive educational beliefs about mathematics (self-efficacy in teaching math, internal locus of control, and availing epistemic beliefs). This supports that a relationship exists between teachers' PCK and their beliefs (Desimone, 2009). Since PCK involves knowledge of content and students, knowledge of content and teaching, and knowledge of curriculum (Hill et al., 2008), improving these aspects would yield more adaptive beliefs relating to mathematics knowledge for teaching.

In terms of antecedents of teachers' educational beliefs, years of experience in mathematics teaching emerges as positively associated with self-efficacy beliefs in teaching mathematics at the onset of the PD program. This finding is expected and consistent with previous research (Wolters & Daugherty, 2007), as one would assume that more experienced teachers are likely to know more about teaching and the content they teach, and in turn, feel more confident in successfully performing mathematics teaching related tasks.

Teachers' mathematical background, specifically the number of mathematics hours taken at college, can serve as a moderator in the extent that teachers enhance their self-efficacy in teaching mathematics throughout a PD program. In other words, teachers who enter the program with less college mathematics hours are more likely to grow in mathematics teaching self-efficacy compared to their counterparts who have more college mathematics hours. This finding suggests that having a strong background in math content plays a role in teachers' beliefs about their ability to be effective teachers (Stevens et al., 2013). The practical implications for PD programs include providing more support and scaffolding to teachers who lack a strong background in the subject matter they teach so that their PCK, and in turn, self-efficacy for teaching mathematics grow (Desimone, 2009). However, future studies are needed to elucidate which aspects of PD enhance various types of educational beliefs among mathematics teachers.

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TECHNOLOGY IN INTERMEDIATE ALGEBRA: RELATIONSHIPS WITH ANXIETY AND OPPORTUNITY TO LEARN

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This study focuses on the role technologically enhanced learning environments (TELE) and assessments have on mathematics anxiety and the opportunity to learn for first year college students in Intermediate Algebra. Qualitative data revealed that technology in the classroom did not improve mathematics anxiety for all students; in fact, TELE may increase student anxiety when seen as rigid and impersonal. The technological environment utilized –MyMathLab– focused on procedural knowledge, and this was valued by students. Future research is needed to understand how conceptual understanding of mathematics can be enhanced and interpersonal relationships can be encouraged in TELE.

The American higher-education system has embraced technology, with a trend towards more online content (Bell & Federman, 2013). The acceptance of technology in the classroom seems complete. However, there is little empirical data concerning the relationships between technology in the classroom and affective states of students (D'Mello, 2013). Many first year college students in the United States are unprepared for general education College Algebra requirements (Rakes, Valentine, McGatha, & Ronau, 2010) and, therefore, must enroll in Intermediate Algebra. At many universities this type of course typically utilizes technologically enhanced learning environments (TELE) for much of the course work and assessments (Epper & Baker, 2009) and, therefore, research is needed to understand how this affects students' math anxiety and opportunity to learn.

Galla and Wood (2012) found a significant negative correlation between mathematics achievement and mathematics anxiety. Jameson and Fusco (2014) found that developmental courses gave students mastery experiences and that having students work with useful math outside of school improved self-efficacy and reduced anxiety. Bong, Cho, Ahn, and Kim (2012) also found a similar inverse relationship between mathematics anxiety and self-efficacy, and Bandura (2012) found it irresponsible to diminish self-efficacy, as this may exacerbate anxiety. ChanMin, Seung, and Cozart (2014) found interactions with instructors and peers decreased anxiety for online mathematics students. In a meta-analysis of 24 studies, D'Mello (2013) revealed discrete affective states associated with boredom, engagement, and confusion were frequently related with the use of technology in the classroom; curiosity, happiness, and frustration were less frequently related; and anxiety was infrequently related to the use of technology in the classroom. D'Mello suggested that more interactive software may lead to better engagement, more positive feedback may lead to greater happiness, and technology-based high-stakes assessments may lead to more frequent relationships with anxiety.

"Opportunity to learn is considered the single most important predictor of student achievement" (National Research Council, 2001, p. 334). Incorporating TELE changes how math is presented and the types of mathematics proficiencies assessed. Hiebert and Grouws (2007) state, "different kinds of teaching facilitate different kinds of learning" (p. 380), and how material is presented affects one's opportunity to learn. Brophy (1999) states that concepts and connections need to be openly discussed and presented in a coherent, structured manner. Much of the opportunity to learn rests on curriculum and resources used for instructional delivery, but it is not just exposure to the material that is important. The material must build upon previous knowledge, it must be accessible to the student, and the environment must be supportive. To promote conceptual understanding and not just procedural knowledge, concepts and connections to mathematical facts should be explicitly taught (Gamoran, 2001; Hiebert, 2003; National Research Council, 2001), students should be allowed to struggle (Hiebert and Grouws, 2007), and students should be required to provide explanations and analyses of the mathematical content (Hiebert & Wearne, 1993). Gresalfi and Barab (2011) find an integral role for the teacher in promoting conceptual awareness.

This study asks, what are some of the relationships between mathematics anxiety and student perceptions associated with TELE, what are some of the relationships between opportunity to learn and student perceptions associated with TELE, and what interactions exist between anxiety and opportunity to learn in a section of Intermediate Algebra?

Method

Context

An Intermediate Algebra course offered in a ten-week summer session at a research-based southeastern United States public university was the setting for this study. The face-to-face aspect of the course was located in a classroom with individual computers for each student, a Sympodium by Smartboard technologies, a dedicated computer for instruction, and a large white board mounted behind a pull-down projector screen. An instructor lectured and facilitated most discussion, a teaching assistant participated as a tutor and part-time lecturer, and the principal investigator of this study sat in as an observer. During class, video and PowerPoint presentations were utilized, clicker assessments were employed, and a white board was used for notes. All students were also required to attend two hours a week outside of class in a computer lab located elsewhere on campus where they completed homework and quizzes in MyMathLab by Pearson. This computer lab was open several hours per day, it was closed on the weekends, and it was supported by graduate assistants hired as tutors.

Participants

Fourteen students were enrolled in the course. All were asked to participate in an exploratory study on the effects of using computer-based assessments in a mathematics classroom. Out of the 14 students, nine gave informed consent. At the beginning of the semester several of the participants stated a desire to major in mathematics intensive subjects such as engineering, and several stated they were taking this course for the second time. All students were informed that their participation was voluntary and would not affect their grade.

Procedure

The mixed methods approach was conducted in four parts. First, the principal investigator observed the classroom. Second, students completed a student anxiety survey consisting of 26 items about their anxiety towards mathematics and computer technology. Fourteen items were from the Mathematics Anxiety Scale-Revisited (MAS-R) by Bai, Wang, Pan, and Frey (2009), and twelve were from the New Computer Anxiety and Self-efficacy Scales developed by Barbeite and Weiss (2004). This survey employed a five-point Likert-type scale and was administered towards the end of the semester. The third part of the study had participants provide responses to a questionnaire about their experiences with the computer-based learning environment MyMathLab. This was administered the day after the survey, and the items were:

1. To what extent does MyMathLab help you learn mathematics?
2. Which tasks or topics, if any, do you feel comfortable working on using MyMathLab?
3. Which tasks or topics, if any, would you rather complete using paper and pencil?
4. How did the use of MyMathLab for homework differ from traditional paper and pencil homework?
5. What are strengths of using MyMathLab to assess your learning of mathematics?
6. What are limitations of using MyMathLab to assess your learning of mathematics?

7. To what extent is MyMathLab user friendly? Explain.
8. How comfortable were you with the computer before using MyMathLab?
9. Has your attitude toward technology changed since using MyMathLab?
10. If given the choice would you take another web-based mathematics course?

Finally, the results of the student anxiety survey were used to select students with the highest self-reported anxiety, and these students were asked to participate in follow-up interviews to clarify questionnaire responses. For the qualitative data, thematic analysis was employed and data representing emergent themes were reported.

Results

Although many students self-reported high levels of mathematics anxiety, most participant responses to the student questionnaires and interviews revealed a positive overall disposition to the classroom experience. However, several students also expressed a negative disposition to the use of technology for instruction and towards the computer lab mandate.

The student anxiety survey administered to the nine participants had a Cronbach's alpha of .87 for mathematics anxiety and .69 for computer anxiety, and a mean mathematics anxiety of 3.22 and a mean computer anxiety of 2.39.

Mathematics Anxiety

There was evidence that the use of technology in the classroom was negatively related to mathematics anxiety. The Smartboard Sympodium was only used sporadically, and on one occasion a female student experiencing math anxiety stated, "It is easier to take notes when you write on the board." The use of videos in the classroom was also related to mathematics anxiety for some; as this same student responded, "I would rather have a person explain it to me." She almost came to tears one day when expressing that because of previous mathematics experiences and the College Algebra general education requirement, she was scared to attend university.

There was also evidence that timed computer-based assessments and the mandate requiring attendance in the computer lab were related to anxiety. Another student spoke of two issues regarding the computer-based assessments. During her interview, she questioned why the computer lab was not open more often. Her anxiety associated with timed computer-based assessments and assessments incorporating clickers was also evident on her questionnaire. In response to question six, she answered, "I hate timing stuff, a 15 minute quiz is not appropriate, that's a joke," and her response to question ten was, "Sure but no timing quizzes please! I know there are many fast thinkers but some of us need to process and work out our problems."

During another student's interview, a question was asked about why the use of computers in this course did not help with his anxiety. He noted mathematics was painful for him in the best situations and not having a human to help made him feel isolated. When asked to expand on his answer to question three from the questionnaire, he stated the computer lab was not open enough and having to be in the lab as part of his grade was an unnecessary burden. Although his dissatisfaction seemed evident, when asked to expand on an answer to question ten from the questionnaire he backpedaled. On the questionnaire he wrote, "No, paper is much better," but during the interview he said it did not matter. Mathematics was the problem, not the computer. Whether the class was paper-based or computer-based, he believed it would still be painful because mathematics was the source of his anxiety.

Opportunity to Learn

Some evidence revealed that MyMathLab was related to students' opportunity to learn mathematics, with procedural knowledge being of utmost importance. Many answered the first question of the questionnaire with responses similar to, "I find it helpful to give me step by step instructions on how to do the problems," and six of the nine participants stated MyMathLab was helpful because it accurately demonstrated procedures and algorithms. Quizzes and tests were also administered using MyMathLab, and similar to the homework, these focused solely on procedural knowledge.

Some evidence also revealed students were concerned with inputting correct answers. Several complained their answers were graded incorrectly because of difficulties with formatting. One student responded to questionnaire question six with, "input errors: sometimes the directions are not clear enough about what form or how many significant digits the answers should have," and another responded, "some examples skip over steps plus some of the icons for symbols are missing." Several answered question four with the greatest benefit of MyMathLab being they immediately knew if their answer was correct. One student responded, "This is likely because of the immediate feedback & ability to correct problems by working a new one." Another answered, "With the help offered on MyMathLab is better than checking on the book and work my problems." One particularly insightful response to question seven was "It is self-explanatory and it guides you on how to do the tasks."

Discussion

Overall, student anxiety was associated with math more than technology, and technologically enhanced learning environments and assessments did not necessarily improve the situation. Some evidence revealed personal interactions and interpersonal discourse reduced anxiety, and mandated computer-based assessments along with inflexible scheduling may be associated with increased anxiety. By requiring students to rely on technology instead of human companionship, their anxiety may have been exacerbated.

The type of computer-based assessments also had a relationship to students' opportunity to learn mathematics. Many students enjoyed using MyMathLab, as it helped them demonstrate procedural knowledge; however as the computer-based assessments did not focus on conceptual understanding, students may have obtained the impression that procedures and correctness were more important than conceptualization of mathematical content. Therefore, benefits to anxiety and engagement based on a reliance on procedural knowledge may have been at the expense of the opportunity to learn and appreciate mathematical concepts.

Implications

Future quantitative studies are needed with utilization of TELE being a manipulated variable as well as studies showing differential changes to anxiety over the duration of a semester. These could provide data on the effect of TELE on anxiety. Future qualitative studies are also needed that focus on how to facilitate personal relationships within a TELE and on how inter-personal discourse can be used to reduce anxiety and increase the opportunity to learn. It is imperative for software developers to find ways to incorporate technology that supports conceptual understanding of mathematical concepts. Additionally, teachers should reflect on issues related to procedural and conceptual tasks and students' overall conceptions of what it means to be doing mathematics. Finally, researchers need to promote instruction of concepts and connections and discourse aimed at reducing feelings of separation and anxiety associated with technologically enhanced learning environments.

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CHARACTERISTICS OF DIFFERENT LEARNING ENVIRONMENTS IN GEOMETRY CLASSROOMS

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This study used in-depth interviews of selected teachers to collect qualitative data to address the following research question: What characterize different learning environments in geometry classrooms? The main findings include: Since the experimental group teachers used dynamic geometry software to facilitate investigations, they were able to produce quality conjectures faster. However, as to proving, teachers varied considerably. Some could generate correct proofs, mostly for relatively simple geometric problems, some were able to work out parts of a proof but had difficulties to put the parts together, and the others were very weak in proofs.

To address the crucial need to improve geometry teaching and learning in our classrooms, we conducted a four-year research project – *Dynamic Geometry (DG) in Classrooms*. The basic hypothesis of the project was that use of DG software to engage students in constructing mathematical ideas through experimentation, observation, conjecturing, conjecture testing, and proving results in better geometry learning for most students. The project tested that hypothesis by assessing student learning in 64 classrooms randomly assigned to experimental and control groups. Teachers in both DG and control groups received relevant professional development. Data were analyzed mainly by appropriate Hierarchical Linear Modeling methods.

The study reported in this paper was a part of the four-year research project. The main purpose of this study was to answer the following research question: What are the characteristics of the learning environments in the DG and control groups with regard to conjecturing and proving? We will focus on teachers as learners in this paper.

The DG Approach and Related Literature

Research suggests that alternatives to traditional instructional approaches can be successful in moving students toward meaningful justification of ideas. "In these approaches, students worked cooperatively, making conjectures, resolving conflicts by presenting arguments and evidence, proving nonobvious statements, and formulating hypotheses to prove. Teachers attempted to involve students in the crucial elements of mathematical discovery and discourse" (Battista & Clements, 1995, p.50). These were exactly the elements experienced by the DG group teachers in the professional development (PD) workshop.

Dynamic geometry is an active, exploratory study of geometry carried out with the aid of interactive computer software such as the Geometers' Sketchpad (GSP) (Jackiw, 2001). The instructional approach for using DG software to facilitate students' learning is referred to as the DG approach in this paper. Many researchers conducted studies on using the DG approach in geometry learning. Vincent (2005) found that the DG motivating context and the dynamic visualization fostered conjecturing and intense argumentation; and that the teacher's intervention was an important feature of the students' augmentations—prompting the students to furnish justifications for their statements and checking the validity of their justifications. Hollebrands (2007) identified different purposes for which students used dragging, the main feature of the DG software, and different purposes for which students used measures. These purposes appeared to be influenced by students' mathematical understandings that were reflected in how they reasoned about the physical representations, the types of abstractions they made, and the reactive or proactive strategies employed. Baccaglini-Frank and Mariotti (2010) presented "a model describing some cognitive processes that can occur during the production of conjectures in dynamic geometry and that seem to be related to the use of specific dragging modalities" (p. 225) and used it to analyze students' explorations of open problems. Thus, *w*hen used as a cognitive tool, DG technology can facilitate students' exploration and investigation activities, promote their conjecturing, verifying, explaining, and logical reasoning abilities, and enhance their conceptual understanding of important geometric ideas. However, very few studies concentrated on whether there are different characteristics of the DG and non-DG learning environments, and if so, what they are and how we can conceptualize strategies for dealing with the differences.

Methodology

To gather evidence regarding the characteristics of the learning environments of the two treatment groups (DG and control), we used in-depth interviews of teachers to collect qualitative data. A stratified random sampling method was used to select 12 teachers (six from each of the two treatment groups; stratified based on mathematical abilities and implementation fidelity). Each interviewee was interviewed a minimum of three times during the school year. Protocols for semi-structured clinical interviews (Goldin, 1997) were created. In each protocol, the interviewee was given an activity involving the posing of conjectures from an exploration of a geometric situation and testing of such conjectures to confirm generalization of the findings. Proving was then requested. When interviewing, the researcher watched the interviewee work throughout the activity and asked questions to help uncover his or her thought process and problem solving strategies. All interviews were video recorded and the videos were transcribed.

Data Analysis

The case study (Stake, 1995) method was used in the data analysis. Four teachers with a range of mathematical abilities were selected for the case studies, with pseudonyms Dan, Greg, Chris, and Nancy respectively. Data derived from the interview transcripts were analyzed inductively to formulate categories and themes to describe the phenomenon under study. This process allows for the data segments to be categorized by a system that is derived from the data themselves. A cross case analysis was also be conducted to address the research question.

Descriptive Analysis and Findings

Based on the related literature cited earlier, we identified the following components of Conjecturing and Argumentation: 1) Construction (Constructing the problem situation); 2) Investigation (Investigating the problem situation to generate conjecture); 3) Stating conjecture 4) Testing conjecture, and 5) Proving. What follows is a descriptive analysis (Koedinger, 1998) of the four teacher interviewees' performances within each of the five components. Due to the page limit of this paper, we will mainly discuss the problem given in the third interview:

A 4 by 4 picture hangs on a wall such that its bottom edge is 2 ft above your eye level. How far back from the picture should you stand, directly in front of the picture, in order to view the picture under the maximum angle?

Dan and Greg's Performances

Construction (Constructing the problem situation): Dan was an interviewee selected from the DG group. He was very efficient in constructing problem situations. When given the maximum angle problem, he proficiently constructed the problem situation using GSP tools in the following steps (Figure 1): (1) constructing a vertical segment AB, using the DG measuring tool to make it 4 cm long, representing 4 feet long (even though the 4cm length was not necessary, it was a good idea); (2) using dilation to construct segment AB' = 6 ft (so BB' = 2 ft); (3) constructing a line through B' and perpendicular to segment AB, which is "eye level"; and (4) constructing a free point (labeled "You", simplified as Y) on line "eye level" and segments AY and BY. Then ∠AYB would be the viewing angle. Greg is another interviewee selected from the DG group. In the teacher content pretest administered at the opening PD session of the larger project, his score was among the lowest. At all PD sessions offered in project year 2, when picking up GSP skills, he usually needed extra help from the PD facilitators or peer participants. However, in all three interviews conducted in project year 3, similar to Dan, he showed high efficiency in using GSP software to construct the given problem situations including the situation of the maximum angle problem.

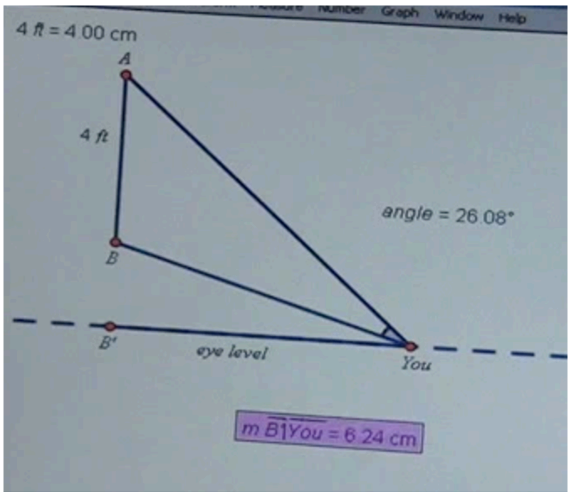

Figure 1. A problem situation

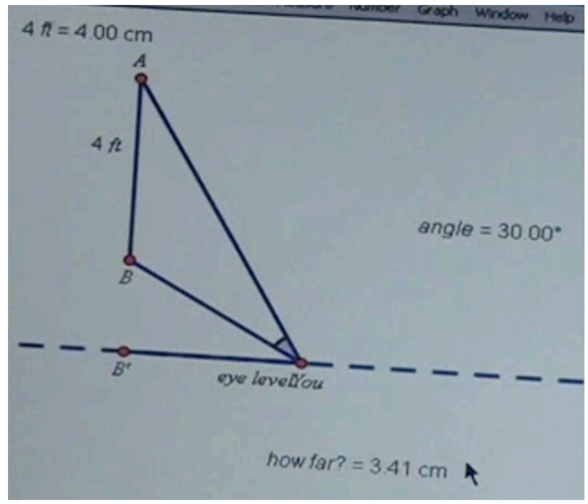

Figure 2. Investigating the problem situation

Investigation (Investigating the problem situation to generate conjecture): In Figure 1, since Y is a free point on line "eye level", Dan dragged it back and forth on the line and found ∠AYB became smaller when point Y moved to right, and became larger when point Y moved to left but after passing a particular location became smaller again. By dragging and comparing the dynamic measurements, Dan found that "particular" location shown in Figure 2 (Y: 3.41 cm [by measuring] representing 3.41 feet away from point B'), where ∠AYB is a 30° angle, which is the maximum viewing angle. To illustrate this fact and formulate a conjecture, Dan continued to explore the situation shown in Figure 2. By guessing ΔAB'Y being a 30°- 60°-90° triangle when ∠AYB is a 30° angle and doing calculations using tan 30°, Dan came up with a conjecture. Greg experienced almost the same investigation process as Dan.

Stating conjecture: As the result of the investigation, Dan wrote his conjecture on his laptop screen: "You should be a distance from the painting equal to the total above eye-level height times tan 30°, which is 6*tan 30° = $6*\sqrt{3}/3 = 2\sqrt{3}$ (feet)." Greg stated the same conjecture using a slightly different language.

Testing conjecture: Dan dragged the free point Y (i.e., point You) back and forth on line "eye level" and always observed the viewing angle AYB being maximized (= 30°) when Y was a distance of $2\sqrt{3}$ (≈ 3.46) feet from point B'. This made him feel positive that his conjecture was correct. Greg did a similar drag test for the conjecture that he stated.

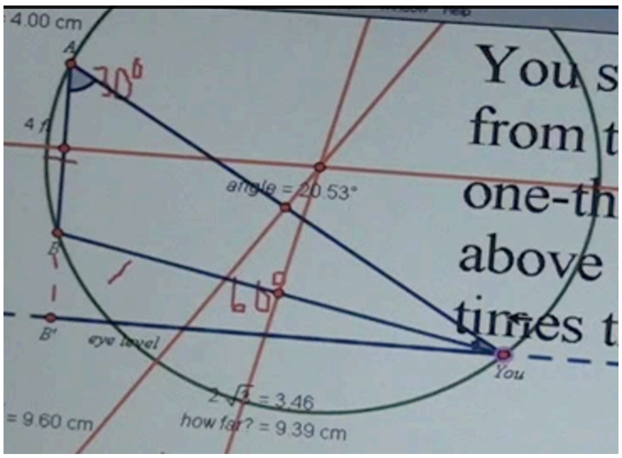

Figure 3. The circle going through A, B, and Y

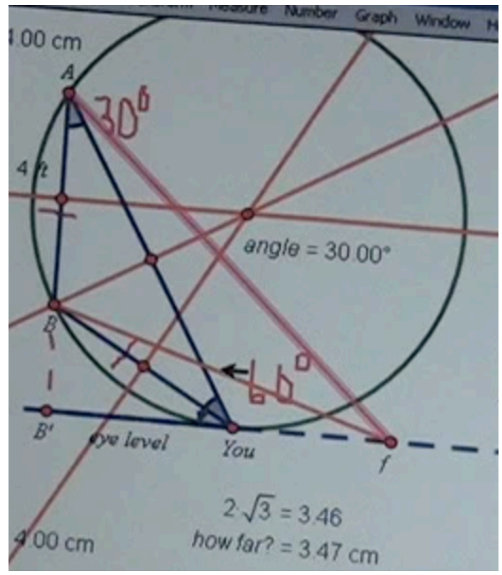

Figure 4. The situation constructed for proving

Proving: For simple proofs such as proving the conjecture that was made for the first interview ("*If two altitudes are equal in length in a triangle, then the triangle is isosceles*"), Dan was able to figure them out correctly in a relatively short time. For more complicated proofs, if time allowed was limited, he might need some help. Dan received such help during the process of developing a proof for the maximum angle problem. He had first tried several different ways to do so but was not able to proceed successfully. Due to the limited time of the interview, the researcher (interviewer) decided to intervene, "Let's construct the circle going through points A, B, and Y (see Figure 3). It might help." Dan constructed the circle by first constructing two perpendicular bisectors on segments BY and AY (Figure 3). Then he dragged point Y back and forth on line "eye level" again. By observing the constructed circle that kept changing size (Figure 3), he found that the viewing angle AYB would become maximized (= 30°) when the circle was tangent to line "eye level" with point Y as the point of tangency (Figure 4). He wrote a different version of his conjecture on his laptop screen: "The maximum viewing angle occurs when you stand at the point of tangency created by line “eye level” and Circle ABY.” Based on another hint from the researcher, Dan constructed a free point (f [should be “F”]) on line “eye level” (Figure 4), and constructed segments Af and Bf. After thinking for a while, he discovered the most important idea in finishing the proof: “It’s related to arcs and inscribed angle somehow. This outside angle [∠AfB] is half of this arc [pointing to minor arc AB] minus this arc [pointing to the small arc that is “inside” ΔAfB and very close to point f]. This is subtracting something, so it is smaller than this angle [pointing to ∠AYB], which is half of this arc [pointing to minor arc AB again].” He measured the two angles using the GSP measuring tool, and dragged point f back and forth to observe the change of ∠AfB, its changing measure, and the fact that m(∠AfB) was always smaller than or equal to m(∠AYB). He stated, “I see. That’s brilliant. I will write the whole process of the proof.” As to Greg, the situation was different. Even for a simple proof such as the one required for the first interview problem, he needed hints or more significant help. For the maximum angle problem, he experienced considerable difficulties, which were related to his weak mathematical background. For example, he did not know how to construct the circle going through three non-linear points.

Chris and Nancy’s Performances

Construction (Constructing the problem situation): Chris and Nancy were from the control group. Chris used a ruler and a pencil to construct the situation of the maximum angle problem (Figure 5). Nancy did not perform any initial constructions. During her investigation of finding the maximum angle, she constructed a situation similar to Figure 6.

Figure 5. The problem situation by Chris

Figure 6. The investigation that Chris did

Investigation (Investigating the problem situation to generate conjecture): Since no free point could be made on line “eye level” as only static tools (a ruler and a pencil) were used, the viewing angle constructed in Figure 5 was only one example. Chris added four more examples (of the viewing angle), and measured all five examples using a protractor (see Figure 6). The measures of the five angles were approximately 28°, 29.5°, 29°, 27°, and 26° (from left to right). Then he estimated the range of the maximum angle, "Basically, around here to here [pointing to points G and J] is the optimal spot. I guess where is being maximized. Somewhere around 29 or 30 degrees." The researcher prompted him by asking if he noticed any special triangles. Chris responded, "ΔCED [E being a point between G and J] might be an isosceles triangle." After measuring the angles of that triangle, he said, "Yeah, it's going to be isosceles, 120, 30 and 30." It took Chris quite a long time in the investigation. Nancy' investigation process was similar.

Stating conjecture: With a hint given by the researcher, both Nancy and Chris conjectured that the maximum viewing angle was at the point of tangency on line "eye level" with circle H (a circle formed by "your eye", the bottom of the picture frame, and the top of the frame).

Testing conjecture: Neither Chris nor Nancy conducted the conjecture testing activity before working on their proofs.

Proving: Both Chris and Nancy were able to do simple proofs, but they had difficulties in doing more complicated proofs without prompting. For the maximum angle problem, without the researcher's hint on constructing a circle mentioned above (i.e., Circle H), they would be unsure how to begin their proofs. The difference between these control group teachers and Dan was that after receiving help, while Dan achieved conceptual understanding of the proof ideas, Chris and Nancy were still focused on the actual measurements in the figure to help with their proofs, which caused some confusion on developing a general formal proof.

Discussion

The above descriptive analysis reveals that when teachers are using DG software to explore geometric concepts and problems, the software's dragging and dynamic measuring features provide great convenience and efficiency for teachers to construct and investigate problem situations. The infinitely many examples or counter examples generated and the dynamic visualization available in the investigation processes help teachers (e.g., Dan and Greg) clearly see what is invariant when other objects kept changing, so as to develop and test conjectures with sound understanding that is necessary for teachers to further create proof ideas. Influenced by the PD designed for the DG group, the teachers have developed a new learning style - conducting problem solving through a learning process characterized by constructing the problem situation – investigation – making conjecture – testing conjecture – proving. The larger project has provided evidence of the effectiveness of this learning process. The situation in the control group is different. Although some teachers recognize the importance of conjecturing and proving, the limitations of using static tools (motionless, time consuming, etc.) in constructing and investigating problem situations do not support the new learning style. For example, Chris and Nancy came up with their conjectures much more slowly than Dan and Greg, and so didn't find time to do the conjecture testing activity.

From the descriptive analysis, we also learned that after our PD sessions, the teachers' reasoning and proving abilities were improved at a slower pace in comparison to conjecturing. Some teachers could generate correct proofs, mostly for relatively simple geometric problems, some were able to work out parts of a proof but had difficulties to put the parts together, and the others were very weak in proofs. Therefore, it is by no means easy to really increase teachers' mathematical reasoning abilities. It would be a long-term task to develop effective strategies for achieving this goal. Furthermore, to take full advantage of the dynamic features of GSP to verify whether a conjecture is true before using it in the reasoning process is an important learning habit, which many teachers did not have. We should spend enough time and energy to help teachers develop this habit.

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PRECALCULUS STUDENTS' ACHIEVEMENT WHEN LEARNING FUNCTIONS: INFLUENCES OF OPPORTUNITY TO LEARN AND TECHNOLOGY FROM A UNIVERSITY OF CHICAGO SCHOOL MATHEMATICS PROJECT STUDY

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Use of technology can help students with their conceptual understanding of functions as supported by national recommendations. This paper reports the technology students had access to when solving function items, the technology features they used, and their relation to achievement. Results are based on a secondary analysis of data from the field trial of the UCSMP Precalculus and Discrete Mathematics (3rd Ed.) curriculum in which 270 students in 14 classes in 6 schools participated. Students' use of technology features had a significant impact on their achievement when solving function items after controlling for opportunity to learn.

The concept of function is one of the most important topics in secondary school mathematics in the U.S. Just as developing a sense of numbers is the goal of the elementary curriculum, developing a sense of functions should be the goal of the secondary curriculum (Eisenberg, 1992). Data from large-scale assessments show U.S. students struggle when solving function problems (Center, 2004; Martin, Mullis, & Chrostowski, 2004). In particular, students have difficulty with different representations intrinsic to the different facets of functions. Each representation (equations, graphs, tables, & words) offers information about aspects of the concept but does not describe it completely (Duval, 2006; Gagatsis & Shiakalli, 2004).

Many experts advocate the best way to raise student achievement is through a curriculum that emphasizes conceptual understanding, problem solving, thinking, reasoning, use of multiple representations, integrated use of technology, and real-world applications and that deemphasizes memorization of rules and procedures (Eisenberg, 1992; McLaughlin & Shepard, 1995; Senk & Thompson, 2003). Such a curriculum is reflective of one recommended by the National Council of Teachers of Mathematics ([NCTM] 2000) and more recently in the Common Core State Standards for Mathematics ([CCSSM] NGA Center for Best Practices and CCSSO, 2010).

There has been little research on strategies students use when solving function problems with or without calculators, including those calculator features students use when graphing calculator technology is available. Even fewer studies have examined these variables within the context of a curriculum based on NCTM's (2000) recommendations or those of the CCSSM, which includes the use of multiple representations and technology integration within the curriculum.

The purpose of this paper is to provide insights into two research questions.

1. What calculator features/strategies do Precalculus students use when solving function items when they have access to graphing calculators with or without computer algebra systems (CAS)? In particular, in what ways do students use these features when using a graphing calculator to solve function items?
2. How is Precalculus students' achievement in solving function items related to their use of calculator features/strategies and their opportunities to learn functions?

Methods

Data to address the research questions are drawn from an evaluation study of the field-trial version of the University of Chicago School Mathematics Project (UCSMP) *Precalculus and Discrete Mathematics* ([*PDM*] 3rd Ed.) (Peressini et al., 2007). The field trial was conducted during the 2007-2008 school year in 6 schools that represented a mix of urban, suburban, and rural environments across the US. There were 270 students in 14 classes, with students in 11 classes using the 3rd Ed. of *PDM* and students in 3 classes using the 2nd Ed. Although the content of the two editions was roughly comparable, differences existed in technology integration and expectations for technology use, with students in 3rd Ed. classes expected to use CAS-capable graphing calculators on a regular basis. To facilitate such use, CAS-capable calculators were loaned to 3rd Ed. classes in sufficient quantity to be loaned to students for use at home and in school. Students in 2nd Ed. classes were expected to have regular access to graphing calculators, generally without CAS.

Prerequisite knowledge was assessed via two multiple-choice pretests, with calculators permitted on pretest 2. Achievement during the year was assessed via two multiple-choice posttests (calculators permitted on posttest 2) and one constructed-response test on which calculators were also permitted. This paper reports results for a subset of function items from posttest 2 for which the Chronbach's alpha was 0.57, which is an acceptable value because the test was designed as a formative measurement (Edwards, 2011). For further details about the assessments, see Hauser (in preparation) or Thompson and Senk (in preparation).

At the completion of posttest 2, students completed a calculator usage form on which they identified the type of calculator available (graphing only or CAS capable) and also identified the calculator features used in solving each problem, with options of a) did not use the calculator, b) used only for arithmetic, c) used graphing features, d) used CAS features, and 5) other.

Data were collected from teachers about the extent to which students had an opportunity to learn (OTL) the content of the course, including the lessons taught and problems assigned for homework. In addition, for each posttest item teachers indicated whether they taught or reviewed the content needed for their students to answer the items. These data assess the extent to which students had an opportunity to learn the function content of the curriculum.

Findings

We discuss students' opportunity to learn functions, their access to CAS or non-CAS technology, the calculator features they reported using on function items, and the extent to which they were successful when using those features.

Students' opportunity to learn functions

In the 3rd Ed., there were 55 function lessons out of 97 lessons in the textbook; teachers reported teaching approximately 79% of these lessons (min = 71%; max = 98%). In the 2nd Ed., there were 55 function lessons out of 106 lessons; teachers reported teaching approximately 90% of these lessons (min = 82%; max = 98%). From the function lessons they taught, 3rd Ed. teachers assigned from 52% to 94% of the possible function homework questions; 2nd Ed. teachers assigned from 55% to 59% of the comparable homework questions.

Among the 16 function items out of 25 on posttest 2, 3rd Ed. teachers reported teaching or reviewing the content for 81-100% of the items. Teachers using the 2nd Ed. reported comparable percentages of 94-100% of the items. Because of space limitations, results are reported only for those function items on posttest 2 for which all teachers reported their students had an opportunity to learn the content to answer the items. Thus, the results reported here are controlled for OTL as reported by the teachers.

Students' use of technology when solving function problems on posttest 2

Table 1 reports students' technology access on posttest 2. In the 2nd Ed. classes, only two students reported access to CAS capable calculators. There was more variability relative to technology access among the students in the 3rd Ed. Classes, with students reporting a mix of both CAS and non CAS capable calculators in many classes. On survey instruments not discussed here, both teachers and students generally reported daily use of their respective graphing calculator technology.

Table 1. *Number of students who had access to type of graphing calculator by class*

	3rd Ed. Classes											2nd Ed. Classes		
	410	411	414	415	416	418	419	420	421	422	423	412	413	417
No CAS	3	3	5	19	13	0	0	4	4	6	4	22	19	19
Had CAS	13	17	12	0	1	18	11	15	20	19	23	1	1	0

Note. Number of students using CAS capable calculators $n = 151$.
Students who did not have CAS used typical graphing calculators.

Table 2 reports the calculator features students reported using on those posttest 2 items designated as *technology neutral*, meaning students could have used a calculator to solve the item but could have easily solved it without the calculator, as well as the percent of students using that strategy who obtained a correct solution. On three of the four items for which all teachers reported students had an opportunity to learn the content, a Fischer's exact test indicated significant differences in achievement between students who used any strategy compared to students who used no strategy, with the students who used a strategy scoring higher. With the exception of item 38, few students reported using CAS features.

Table 2. *Number of Students Indicating Use of Calculator Features on Technology Neutral Function Items on Posttest 2 and Percent Obtaining Correct Solution: Controlled for Opportunity to Learn*

Item	Access to CAS	Calculator Features/Strategies											
		None		Arith		Graph		CAS		Other		Any[a]	
		N	%	N	%	N	%	N	%	N	%	N	%
31. Given the function h defined by $h(x)=\frac{(2x+4)(x-1)}{(x+2)}$.	Yes	68	51	5	40	60	30	7	29	6	67	78	33
What is the behavior of the function near $x = -2$?	No	69	2	0	0	42	48	N/A	N/A	2	48	44	50
37. Suppose $f(x) = x^{1/2}$. What is the set of all values of x for which $f(x)$ is a real number?	Yes**	67	30	44	34	28	82	4	50	3	67	79	53
	No*	49	39	40	43	23	74	N/A	N/A	1	100	64	55
38. Evaluate $\lim_{x\to 10}\frac{x^2-100}{2x^2-23x+30}$.	Yes **	40	5	23	17	34	24	49	76**	1	0	107	46**
	No	43	19	29	24	32	19	N/A	N/A	6	33	67	22
46. Which of the following could be an equation for the graph at the right? [graph of polar function shown]	Yes **	42	48	4	75	92	85**	10	80	0	0	106	84
	No**	28	57	8	88	77	86**	N/A	N/A	1	0	86	86

Note: Results significant at $\alpha = 0.05$ level are indicated by * and at $\alpha =0 .01$ by **
[a] Any refers to the use of Arithmetic, Graph, CAS, or other strategies and is compared to the use of no strategy (none).
Rows may add up to more than 100% because of students' reported use of multiple features.

There were also five calculator inactive function items on posttest 2 for which all teachers reported having taught or reviewed the content needed to answer the items. *Calculator inactive* items are those in which there is no advantage, or possibly even a disadvantage, to using a calculator. On these items, students generally reported not using a strategy; students who used a calculator on the inactive items, on average, scored lower than those who did not use a calculator (see Table 3).

Table 3. *Number of Students Indicating Use of Calculator Features on Technology Inactive Function Items on Posttest 2 and Percent Obtaining Correct Solution: Controlled for OTL*

Item	Access to CAS	Calculator Features/Strategies											
		None		Arith		Graph		CAS		Other		Any[a]	
		N	%	N	%	N	%	N	%	N	%	N	%
3. For a function g, the derivative at 2 equals -1, that is $g'(2) = -1$. Which of the following describes the meaning of $g'(2)$?	Yes	136	58	4	25	4	0	1	0	0	0	9	11
	No	111	56	1	100	0	--	N/A	N/A	0	0	1	100
4. Refer to the graph of function f at right. On which of the following intervals is f increasing?	Yes	136	82	3	83	3	100	4	100	0	100	10	100
	No	110	94	1	100	1	0	N/A	N/A	1	100	3	67
6. A function h is graphed at right. As $x \to +\infty$, what is true about $h(x)$?	Yes	133	74	4	75	3	67	4	75	0	0	11	73
	No	105	79	1	100	5	80	N/A	N/A	1	100	7	67
2. Use the graph of the function $f(x) = ax^3+bx^2+cx+d$ shown at right. How many real solutions are there to the equation $f(x) = ax^3+bx^2+cx+d = -2$?	Yes	130	74	6	67	4	25	4	75	1	0	15	53
	No	109	78	3	0	0	0	N/A	N/A	1	100	4	25
4. What is the value of $f(g(1))$? [using the graph]	Yes	129	69	6	67	6	67	3	67	1	100	16	69
	No	109	83	1	100	0	0	N/A	N/A	1	100	1	100

Note: [a] Any refers to the use of Arithmetic, Graph, CAS, or other strategies and is compared to the use of no strategy (none).
Rows may add up to more than 100% due to rounding or if students reported using more than one strategy.

Opportunity to learn and use of technology as predictors for achievement on posttest 2

A multiple regression analysis was conducted to evaluate how well use of technology and OTL measures predicted achievement on posttest 2. Analysis of the data verified no outliers;

assumptions related to collinearity, independence, and normality were met. Pretest 2 was used to control for differences in prior knowledge; OTL Lessons (percent of function lessons taught) and DidUseStrategies (number of times a student reported using a calculator feature to solve an item) were used as predictors, with achievement as the criterion variable. Table 4 reports the standardized and unstandardized coefficients and correlations for each variable in the regression model. Approximately 28% of the variance in achievement for posttest 2 can be accounted for by the combination of calculator feature and OTL measures, after controlling for prior knowledge. According to the model, each time a student used a calculator feature/strategy to solve an item his/her overall percent correct on posttest 2 function items went up, on average, 0.54 points; the percent score also went up 0.48 points for every one percent increase in function lessons covered.

Table 4. *OTL Measures and Use of Technology as Predictors of Achievement on Posttest 2*

Variable	Correlations				b	β
	DidUseStrategy	OTL Lessons	Pretest 2	Achievement		
Pretest 2				.415**	.391	.364**
OTL Lessons			.206**	.394**	.484	.318**
DidUseStrategy		.015	-.135	.065	.536	.110**
					Intercept = 1.60	
Mean	4.93	78.09	41.88	58.38		
SD	3.37	10.78	15.27	16.42	$R^2 = 0.283$	

Note: ** $p < .01$; ($F(3, 267) = 35.02$, $p < 0.01$)

Discussion

This paper has focused on different calculator features Precalculus students used to solve function items and how those features/strategies influenced achievement. On technology neutral items when controlling for OTL, students typically used appropriate calculator features, and in general, were more likely to be successful than students who did not use a calculator. For two of the neutral items (31, 46), graphing was the most common strategy used regardless of access to CAS; on item 38 students who had access to CAS used this feature slightly more often than graphing. On technology inactive items, although some students attempted to use a calculator, the majority (93-96%) did not. The data also clearly show students did not indiscriminately attempt to use CAS. Overall, the results are promising and suggest students are using calculator features appropriately when solving function items.

In this study, students reported using graphs to solve function items more often than indicated in past research (Huntley, Marcus, Kahan, & Miller, 2007). Even when students had access to CAS capable calculators, they still often preferred a graphing solution over CAS. Additional research investigating the relationships between achievement, the use of calculator features, and opportunity to learn is needed, especially in the teaching and learning of functions.

Although the results are promising, it is important to note the results of the calculator usage survey are self-reported data from students and, as such, there is no way to determine if students actually used the calculator or strategy as indicated. The regression analysis suggests that opportunity to learn in terms of function lesson coverage potentially mediates achievement as does the extent to which students use their calculators appropriately. Although detailed data were collected about how teachers implemented the curriculum in terms of lesson coverage and homework assigned, more detailed data are needed about the extent to which teachers integrated the use of calculator technology in their instruction. Such data would enable more detailed analysis of class-to-class and school-to-school differences in end-of-year achievement.

Some educators believe that when students are able to move frequently between representations as they solve function problems, they become more aware of the connections between these representations and begin to see how information about functions is presented in different ways and different representations (Kaput, 1989). Kaput (1989) argued dynamic technologies, such as graphing calculators, can be instrumental in helping students understand linked representations as the different calculator features (e.g., tables, graphs) relate to different representations. Students' use of calculator features/strategies in the study reported in this paper is encouraging, perhaps indicating students are becoming more proficient at choosing those calculator features that are most viable in a particular situation. Given the importance of functions in the secondary curriculum of the CCSSM and the results reported here regarding the relation between calculator features and student achievement, deeper understanding of how graphing calculator features with or without CAS can be used to enhance achievement in this topic area is worth further research.

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EARLY CHILDHOOD GENDER DIFFERENCES IN NUMBER SENSE WHEN LEARNING WITH IPADS

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This paper addresses how a set of research-based applications on the iPad impacted early childhood number sense development. The researchers conducted this study with children enrolled in an urban Head Start center. Students were randomly assigned to a treatment or comparison group, evaluated with a pretest, and, after playing iPad games designed to improve number sense for six weeks, were given a posttest to examine the influence of the games on children's number sense. While the mathematics achievement of the children increased during the study, there was no significant difference in performance gains between male and female students.

While it has been shown that young children are ready to learn mathematics in pre-school (Lipton & Spelke, 2006), there is concern that this learning is not supported effectively by the reality of ill-planned semi-academic activities or unprepared pre-school teachers (Stipek, Schoenfeld, & Gomby, 2012; Stipek, 2013). Early mathematics interventions that focus on children foundational mathematics skills are especially critical for children who are at risk for future school failure (Clements & Sarama, 2007).

With early academic gains crucial for further academic achievement (Entwisle & Alexander, 1998), the appearance of the gender gap in mathematics as early as preschool (Levine et al., 1999; Rathbun et al., 2004) is problematic. A means to provide extra support is through the use of technology. Technology has been reported to effectively engage students in school learning (Taylor & Parsons, 2011); specifically, iPads can serve as a platform for young children to learn mathematics (Sherr, 2011).

Because there are limited empirically based guidelines about the implementation of technology for effective learning (Means, 2010), this study provides insight about the use of iPads in an early childhood mathematics setting.

Theoretical Framework

Number Sense Development in Early Childhood

Number sense is a broad academic field without a strict definition, but with numerous examples (Greeno, 1991), among which subitizing, counting, identifying numerical numbers (Arabic number), ordering, and comparing are essential foundations (Clements & Sarama, 2014). Subitizing is "instantly seeing how many" (Clements, 1999, p. 400) and is viewed as a prerequisite tool for counting because children who can count can subitize, but not vice versa (Klein & Starkey,1988).

Learning to count is a complex process that involves movement of manipulatives (Kilpatrick, et al., 2001) to help children form a concrete understanding of quantity. Then, the children learn to assign a verbal number name to the objects being counted. Finally, children learn that the last verbal number name corresponds to the quantity of the set (Jordan & Levine, 2009).

Subitizing also contributes to the development of initial ideas of cardinality for young children (Clements, 1999). Cardinality is "the ability to represent the number of discrete entities in a set and to appreciate the numerical equivalence of all sets whose members can be placed into exact one-to-one correspondence" (Brannon & Van de Walle, 2001, p. 54). Based on the learning of cardinality, children are able to investigate the ordinal relationship of numbers by establishing the *greater than* and *less than* positions of numbers.

Different from comparing numbers, creating equivalence relations and comparing quantities typically begins in infancy, without training in numbers. Children around the age of four can distinguish between quantities without counting. In particular, young children can decipher which stack of blocks has more or less (Jordan & Levine, 2009).

In sum, number sense development in early childhood starts as early as infants and requires effective educational interventions to support effective learning in different sub-areas. This study used iPad games based on the above framework and existing prior designs, as an attempt to assist young children in number sense development.

Research findings indicate that gender differences in mathematics may start as early as in pre-school. For example, differences are seen in the spatial abilities of four-year old children (Levine et al., 1999; Penner & Paret, 2008) and in kindergarten children's general math performance (Rathbun et al., 2004). A study conducted in Australia involving 176 preschool children found that boys performed significantly better in quantitative concepts, while girls performed better in subitising (Howell & Kemp, 2010). Early academic gains are crucial for cognitive development which functions as the foundation for further academic achievement (Entwisle and Alexander, 1998). These studies suggest the need for interventions to narrow the early-age gender differences in learning math. One such intervention is the use of technology; for example, using an iPad to improve mathematics and science learning (Aronin & Floyd, 2013). Specifically, this study addressed the following research question: Are there significant differences in mathematics achievement between boys and girls using iPad mathematics games in an early childhood setting?

Method

Participants

One hundred preschoolers from a large Head Start center in an urban southwestern school district were randomly selected to participate in the study. Fifty students from eight classrooms were randomly assigned to the intervention, and fifty to the comparison group. At the pre-test date, the sample consisted of 50 intervention and 50 control group students. At the posttest date, 45 intervention and 41 comparison children remained.

Procedures

Children assigned to the intervention played the *Math Shelf* games on iPads. Students assigned to the comparison group had the choice to play five of the most downloaded, and best-reviewed pre-K math apps sold in the Apple store. Children in both conditions played three days a week, 10 minutes each session, for six weeks (April 28, 2014 to June 6, 2014). Game play occurred in a separate classroom and was supervised by graduate student researchers.

Intervention Software. *Math Shelf* is a pre-K iPad number sense application that uses a variety of virtual manipulatives, math puzzles, and games to systematically teach eight number sense competencies. Before playing *Math Shelf*, preschoolers take an iPad placement test. This test determines whether children are assigned to 1-to-6 or 1-to-9 number sense activities and games. The eight numbers sense competencies that *Math Shelf* teaches are: connecting number names to quantities (1 to 9), connecting number names to symbols (1 to 9), matching numbers to quantities (1 to 9), ordering numbers and quantities (1 to 9), one to one counting/counting how many in a set (1 to 9), building quantity and number relationships (1 to 9), place value (units, tens and hundreds), and number decomposition.

Control Software. Teachers that incorporate iPads into their instruction, download apps from the App Store. In order to vet apps for quality of instruction and engagement, teachers may rely on reviews of apps published by reputable sources, and the top-grossing app lists in the education category. The comparison condition used both of these sources to select five pre-K math apps for the control group children to play with. We first selected the two top grossing pre-K math apps: (1) *Team Umizoomi* and (2) *Numbers with Nemo*. Next, we chose three of the most widely- and best-reviewed pre-K math apps by Children's Tech Review, Moms with Apps, the Parent Choice Awards, Common Sense Media, and USA Today's Top 10 Apps for Kids. The three best-reviewed apps were: *Monkey Math*, winner of the Parent Choice Award in 2013, and five star rating from Common Sense Media. *Elmo Loves Math*, winner of USA Today's Top 10 Apps for Kids and the Parent Choice Award in 2014. *Park Math* HD, which won the 2013 Children's Tech Review Editors Choice Award. These apps taught various pre-K and Kindergarten math skills and content.

Test administration procedures. Graduate student researchers tested all children individually on an iPad number sense assessment that provided audio and visual instructions on how to complete each item. All children were pretested during the week of April 21th and post-tested during the week of June 9th. Test administration scripts were strictly followed.

Measures

A 62-item number sense assessment was developed for the study. Children took the untimed assessment on iPads, and completed the test on average in six minutes.

Assessment Reliability. Test-retest reliability was collected on a sample of 20 students (average age 4 years 5 months) from a Head Start center in northern California. These children took the test five days apart in the Head Start office. The test-retest reliability intra-class correlation was 0.97. Cronback alpha inter-item reliability was .94. The number sense assessment measured the following skills:

Results

To examine the baseline balance between the two randomized groups on the number sense assessment, we used two-sample t-tests (for continuous variables) and chi-square tests (for categorical variables). There were no statistical differences between the intervention and comparison groups in terms of number sense pre-test ($t(99)=0.15$, $p=.89$), age ($t(99)=.64$, $p=.53$), gender ($X^2(1)=.16$, $p=.69$), and ethnicity ($X^2(3)=.22$, $p=.97$) assuring that our randomization was successful. Mixed factorial ANOVA was used to test the change before and after the intervention. Students in both conditions, intervention (n=50) and comparison (n=50) improved. Table 1 shows the growth from pre- to post-intervention. There was a significant and sizable effect for the intervention on number sense (Cohen's *d*=0.57, *p*<.001).

Table 1. *Estimated change from pre- to post-intervention*

	Intervention	Control	Group Difference
Pretest	21.5 (12.1)	21.5 (11.5)	
Posttest	34.6 (16.6)	25.5 (11.6)	9.1
Pre-to-Post Change	13.1	4.0	*p*<.001
p-value	*p* < .001	*p* =.005	
95% CI	[10.4, 15.8]	[1.3, 6.8]	[5.3, 12.8]
Effect Size*			.57

- Effect size (Cohen's *d*) was calculated based on observed standard deviation at post-intervention assessment pooled across the intervention and control conditions.

To specifically address the research question, we examined the interaction between growth and gender and found that the interaction was not significant ($F(1, 84)=0.58$, $p=.449$). Females scored 21.0 on the pretest and 28.4 on the posttest, while male students scored 21.9 on the pretest and 31.6 on the posttest.

Discussion

In our study, unlike previous research (Levine et al., 1999; Rathbun et al., 2004, Penner & Paret, 2008), there were no differences in the mathematics achievement gains between boys and girls. Future quantitative analysis should disaggregate the data further, examining the potential gender gap separately for treatment and comparison conditions. The current study provides evidence that all students can learn when provided time and access to iPads. Not surprisingly, students learned more when using iPad applications that were more explicitly connected to research-supported subdomains of number sense.

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EFFECTS OF AN ONLINE ALGEBRA REFRESHER PROGRAM ON STUDENT PLACEMENT EXAM SCORES

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A two week interactive, online summer bridge program was developed and piloted to help incoming freshmen at risk of failing their college algebra course succeed in their first semester. The bridge program was designed to remind students of processes and concepts they may have forgotten over time, as well as to help them develop confidence and study skills. This was accomplished by using a mix of online resources and live tutoring, provided through Blackboard Collaborate. Initial results from the pilot study reflect a statistically significant increase in the participating students' mathematics placement exam score.

The past decade has witnessed a surge in summer bridge programs meant to better prepare incoming students for traditionally difficult courses at universities across the country. Indiana State University (ISU) is a midsized public four-year institution. A majority of ISU's students are first generation college students from both rural and urban areas (Indiana State University, 2013). Many of these students are under-prepared to succeed in college level classes: especially those in mathematics. Freshmen at Indiana State traditionally struggle with College Algebra (Math 115), a required course for many majors with approximately a forty percent fail/drop rate.

For Science, Technology, Engineering and Mathematics (STEM) disciplines in particular, being placed into a lower level mathematics course than expected can cause a delay in graduation for many majors. Low mathematics placement delays enrollment in STEM major courses, causing some students to lose interest and switch to a non-STEM related field (Reisel, Jablonski, Hosseini, & Munson, 2012). Keeping students in STEM fields and increasing those numbers continues to be a high priority in the United States. STEM field graduates are needed in the workforce, but it is these same disciplines "in which American education is failing most convincingly" (Members of the 2005 "Rising Above the Gathering Storm" Committee, 2010, p. 48).

In an effort to improve student performance in the college algebra course, the university's Math and Writing Center, in collaboration with faculty from the Mathematics and Computer Science department, designed a cost-effective online summer refresher program. This program was targeted at students who had either barely missed placing into College Algebra or barely succeeded in placing into that course. During the program, students learned or reviewed prior mathematical content, developed study skills, learned time management, and became familiar with tutoring resources. Though the process of measuring the success of this program is ongoing, this paper analyzes participants' pre and post mathematics placement test scores to determine if participation increased their test scores and in some cases, resulted in the ability to move into a higher level course.

Given budget constraints, an online bridge program was used because it does not necessitate the high cost of housing or providing meals for students. Instead of attending traditional lectures, participants were given a curriculum based on the resources readily available through Khan Academy. Khan Academy, found at www.khanacademy.org, is a not-for-profit website with the professed goal "of changing education for the better by providing a free world-class education for anyone anywhere" (Khan, Salman, 2014).

The participants moved through the modules over the course of two weeks, spending three to four hours daily on their work. Participants had access to live tutors who appeared at set times every day on Collaborate, a synchronous video and interactive whiteboard function of Blackboard. The tutors answered questions and gave students advice on studying mathematics. At the end of the two weeks, students came to campus and re-took the mathematics placement test with hopes of improving their scores. Most wanted to score high enough to move into a course a level above that which they placed into the first time.

This pilot program ran in the summer of 2014, but the university is interested in expanding the program based on positive results. For this reason, it is crucial that stakeholders carefully analyze the program's efficacy in order to determine a) whether students in the program score significantly higher on their placement exams and b) whether these students have a better chance of succeeding in College Algebra. To this end, pre and post test data was analyzed, and grades for students who went through the refresher will be compared to grades of students who did not at the end of the Fall 2014 semester. While preliminary findings are positive, changes to the program will be necessary if it is to be expanded in the future.

Literature Review

This study is framed through Astin's (1999) theory of student development, which states that students are more likely to retain and persist to graduation if they are invested socially and academically in their university. Bridge programs give students the advantage of becoming active in their work before the start of their first semester, exposing them to resources and the technology of their school before they begin classes. Such programs introduce students to their peers, as well as administrators and faculty members, allowing them to find support early on in the semester.

Bridge programs can be just as beneficial for a university as they are for students. Over 40 percent of students attending four year institutions take remedial courses, which is time consuming and costly for all parties (Adams, 2012). If students can avoid remedial courses by brushing up on their skills before school starts, they will be more inclined to sign up for more challenging classes in their first semester (Wathington, Pretlow & Mitchell, 2011). Despite the abundance of bridge models, there is a notable gap in conclusive evidence as to whether these programs benefit students past their first semester in college because there are so few longitudinal studies of such programs (Cabrera, Miner & Milem, 2013; Garcia & Paz, 2009).

A longitudinal study was done at the University of Alabama examining the effect of their Engineering Math Advancement Program, a five week residential program targeting engineering students who tested into pre-calculus but needed to be prepared to take calculus their first semester. The students in the program were followed for three years to see if the program increased retention of students in STEM fields. A twelve percent increase in retention was observed after three years. However, the grades of the participants were lower in their first mathematics course compared to the students who did not participate in the bridge program. One possible conclusion from this study is that a bridge program perhaps should "focus on the goal of improving the students' mathematical knowledge in order to improve their chances of success in their first mathematics course rather than accelerating their program" (Gleason et al., 2010, p.13).

In part because of this lack of information regarding bridge efficacy, Indiana State University chose to pilot an inexpensive, short online refresher model. Though this design is not entirely uncommon, many models are in place at a variety of schools. Some programs offers a multi-week residential model for students who need extra help with math, while other programs allow students to select the academic subject of most concern to them (generally math, reading, or writing) (Doerr, Arleback, Staniec, 2014; Hodara, 2013; Reisel et al., 2012; Wathington, Pretlow & Mitchell, 2011). Some institutions run dual residential and online programs, though results indicate that students participating in residential programs saw greater success than those in online programs (Reisel et al., 2012). Almost all models feature tutoring by students currently enrolled in the university, as they provide a cost-effective labor force and positive role models for nervous students (Adams, 2012; Hodara, 2013).

Methodology

All Indiana State University students must take a placement test as incoming freshmen, and those who scored between a 9 and a 15 inclusive were eligible for enrollment into the program. Students must score a 12 or higher on the placement exam to qualify for College Algebra. Qualifying students were identified by their academic advisors. Students with a 90% complete rate for the program were offered a book scholarship, which, along with a chance to retake the placement exam, motivated students to participate in the program. At the end of the two week program, students retook the mathematics placement exam in a proctored setting. Out of the 28 students who enrolled in the pilot of this program, 22 completed the program while 6 students either dropped out or failed to retake the placement exam. These six were excluded from the data analysis.

Students worked primarily online during the Summer Bridge Program. An online Blackboard course was created where students could access information about the program as well as find links to the Khan Academy Training videos and modules. Students were also required to use the their university email and login to better familiarize them with both the university email, website and Blackboard prior to the beginning of the semester. Under the Collaborate tool, students could have online interaction with tutors if they needed assistance. The tutors were together on campus and were equipped with laptops, headsets, microphones, and electronic writing tablets.

In Collaborate, the students could see work done by the tutor as they used the tablets and they could see and communicate with the tutor in real time. Using Collaborate was an opportunity for students to have a personal connection while doing distance learning. Since students could work at their own pace, tutors were available on Collaborate from 9am-6pm seven days a week with as many as 4 tutors available for assistance. In addition, the blog utility of

Blackboard was used so that the students could interact with each other and the tutors. The students came to campus for the posttest and after taking the exam had a social gathering, where they could meet each other, the tutors, and some of the instructors in person prior to the beginning of the semester.

Results

Students' placement scores from before and after the program were compared as one way to measure the efficacy of the Summer Bridge Program. 22 students finished the program. It should be noted that due to an administrative oversight, two participants were able to participate in the program even though their score was below a 9. In Table 1 below, the initial placement test is the "Pretest" and the retake is labeled "Posttest".

Table 1. *Student placement scores*

	N	Min	Max	Mean	Std.Dev.
Pretest	22	4	11	9.41	2.02
Posttest	22	7	25	13.59	5.24

Performing a 1 tailed paired T-test produced a significant difference in the scores for the pretest ($M=9.41$, $SD=2.02$) and posttest ($M=13.59$, $SD=5.24$) conditions; $p <.001$. The conclusion is that the 14 day Summer Bridge Program had a significantly positive effect on students' placement scores. Three students tested into calculus, moving two courses ahead of their initial placement. Seven students did receive a lower score on the retest than they did on the initial exam. However, none of these students dropped by more than two points and none of them lowered their score enough to be in a lower mathematics course than their initial placement.

At the end of the fall semester, the course grades of the participants will also be compared to a control group to better determine the effectiveness of the program.

Discussion

Several initial issues were discovered during the pilot study. First, as can be seen from the results table, we were unable to successfully recruit students who had already tested into college algebra. Instead, only students who tested into a remedial course were sufficiently motivated to participate in the bridge program. For the next year, we will need to better market the program to those students who currently test into college algebra. Since one of our goals is to increase the pass rate of college algebra, this will be an important issue to consider before next summer.

Another more positive result came from the students who “snuck” in to the program with a lower initial placement score than we intended. Both of those students tested into a more advanced course after their retest. For our particular student population, it may be beneficial to open up the range of allowed scores to include any student who tests into college algebra or below. That would be any score of 20 or below.

It should be noted that the university allows incoming students to take the online test from any location. After the refresher program, students were required to take the test in a proctored setting on campus. In the future, it would be better to have students take the test under similar, controlled environments. Based on historical and anecdotal evidence, it appears that two separate issues may affect initial placement scores. The first is the issue of motivation. Some students do not realize the significance of the placement exam until after they have taken it and attempt to register for courses in their major. They then realize their placement scores will not allow them to take the necessary courses in their first semester, requiring them to either delay their graduation or change their major. The second issue with the placement exam is that the initial test lends itself to the possibility of cheating, which could inflate a student’s score. This could give students false confidence when they enter their first mathematics course and contribute to the high failure rate. Giving the initial test in a proctored setting may alleviate both of these issues.

Finally, while these results are promising, our primary concern is still whether or not the students will successfully complete the college algebra course. We will not be able to determine this until we can compare grades at the end of the semester. We also want to broaden the program next year and collect some additional qualitative data to provide a better overall picture of the students’ experience with the program.

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